TMUA TOPIC TEST

Multivariate Factorisation Topic Test Solutions

Time limit: 40 minutes

Calculators: not permitted

Format: 10 multiple-choice questions

This is a TMUA-style topic test: a timed set of TMUA-style questions, with a particular focus on a given topic. You can also sit this test online at jzmaths.com, where the result will count towards your mastery score.

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Question 1

The positive integers x and y satisfy $xy + 2x + 3y = 42$. How many ordered pairs (x, y) are there?

A 5

B 4

C 6

D 8

Solution 1

Answer: A

$xy + 2x + 3y$ does not factorise, but we can write it as $x(y + 2) + 3(y+?)$. This suggests adding 6 to both sides of the equation:

$$xy + 2x + 3y + 6 = 48 \quad \Leftrightarrow \quad (x + 3)(y + 2) = 48.$$

Since $x, y \geq 1$, we need $x + 3 \geq 4$ and $y + 2 \geq 3$, and both are integers, of course. Therefore, we just need to go through the possible factorisations: 4×12, 6×8, 8×6, 12×4 and 16×3, giving

$$(x, y) = (1, 10), (3, 6), (5, 4), (9, 2), (13, 1).$$

The factorisation 24×2 does not work because $y + 2 = 2$ forces $y = 0$. Hence, there are 5 pairs.

Question 2

For all positive real numbers a and b, the inequality $a^3 + b^3 \geq c\,ab(a+b)$ holds. What is the greatest value of the constant c for which this is always true?

A $\frac{1}{2}$

B 1

C $\frac{2}{3}$

D 2

Solution 2

Answer: B

Notice that on the left-hand side $a = -b$ is a root, so $(a+b)$ is a factor. So factorise the left-hand side: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$. Then dividing the inequality by the positive quantity $ab(a+b)$ gives

$$\frac{a^2 - ab + b^2}{ab} \geq c \quad \Leftrightarrow \quad \frac{a}{b} + \frac{b}{a} - 1 \geq c.$$

For those that are familiar with the AM-GM inequality, you may recognize $\frac{a}{b} + \frac{b}{a} \geq 2$, otherwise, this can be proven.

We know $(a-b)^2 \geq 0$, thus $a^2 + b^2 \geq 2ab$. Since both a and b positive, then so is ab, then dividing by ab gives $\frac{a}{b} + \frac{b}{a} \geq 2$.

Therefore $\frac{a}{b} + \frac{b}{a} - 1 \geq 1$, always, and with equality when $a = b$. The greatest constant that always works is therefore $c = 1$.

Question 3

The expression $a^2(b-c)+b^2(c-a)+c^2(a-b)$ is identically equal to which of the following?

A $(a-b)(b-c)(c-a)$

B $(a+b)(b+c)(c+a)$

C $-(a-b)(b-c)(c-a)$

D $(a-b)(b-c)(c+a)$

Solution 3

Answer: C

Treat the expression as a polynomial in a. Substituting $a=b$ gives $b^2(b-c)+b^2(c-b)+c^2\cdot 0=0$, so $(a-b)$ is a factor. Similarly $(b-c)$ and $(c-a)$ are factors as well.

Now since the expression has degree 3, it equals $k(a-b)(b-c)(c-a)$ for some constant k.

Testing $a=2$, $b=1$, $c=0$ gives the value $4(1)+1(-2)+0=2$, while $(a-b)(b-c)(c-a)=(1)(1)(-2)=-2$, so $k=-1$. Hence the expression equals $-(a-b)(b-c)(c-a)$.

Question 4

Let $a > 0$ and $f(x) = \frac{1}{2}x^4 + ax^3 - \frac{3}{2}a^2x^2 - 2a^3x$. What is the x-coordinate of the local maximum of f?

A $-2a$

B $-a$

C $\frac{a}{2}$

D $-\frac{a}{2}$

Solution 4

Answer: D

Differentiating gives

$$f'(x) = 2x^3 + 3ax^2 - 3a^2x - 2a^3.$$

Notice that $x = a$ is a root of $f'(x)$, so $(x - a)$ is a factor. Hence

$$f'(x) = (x - a)(2x^2 + 5ax + 2a^2) = (x - a)(2x + a)(x + 2a).$$

Therefore the stationary points occur at $x = -2a$, $x = -\frac{a}{2}$ and $x = a$. Since $a > 0$, they occur in the order

$$-2a < -\frac{a}{2} < a.$$

Since $f(x)$ is a quartic with a positive x^4 coefficient, the middle of these three turning points must be the local maximum. Hence the local maximum occurs at $x = -\frac{a}{2}$.

Question 5

Positive real numbers x and y satisfy $x + y = 3$.

For which values of x is $x^3 - 2x^2y - xy^2 + 2y^3 > 0$?

A $0 < x < \frac{3}{2}$

B $\frac{3}{2} < x < 2$

C $0 < x < \frac{3}{2}$ or $2 < x < 3$

D $\frac{3}{2} < x < 3$

Solution 5

Answer: C

First notice that $x = y$ is a root of $x^3 - 2x^2y - xy^2 + 2y^3$, so $(x - y)$ is a factor. Similarly, $x = -y$ is also a root, so $(x + y)$ is a factor. Since the expression is cubic, we can then factorise it completely as

$$x^3 - 2x^2y - xy^2 + 2y^3 = (x - y)(x + y)(x - 2y).$$

Now use $x + y = 3$, so $y = 3 - x$. Since x and y are both positive, we also know $0 < x < 3$.

Substituting $y = 3 - x$ into the factorised inequality gives

$$(2x - 3)(3)(3x - 6) > 0 \quad \Leftrightarrow \quad (2x - 3)(x - 2) > 0.$$

Combining this with $0 < x < 3$, we obtain

$$0 < x < \frac{3}{2} \quad \text{or} \quad 2 < x < 3.$$

Question 6

The equation $2\sin^3 x - \sin x = 2\cos^3 x - \cos x$ has four solutions in $0 \leq x < 2\pi$. What is the sum of these four solutions?

You may use the result $\sin(2x) = 2\sin x\cos x$ for all real values of x.

A 3π

B $\frac{7\pi}{2}$

C 4π

D $\frac{9\pi}{2}$

Solution 6

Answer: C

In a completely shameless exploitation of the multivariate factor theorem, we immediately notice that $\sin x = \cos x$ is a root. Therefore $\sin x - \cos x$ must be a factor. :)

Therefore, moving all terms to one side and factorising gives

$$(\sin x - \cos x)(2\sin^2 x + 2\sin x\cos x + 2\cos^2 x - 1) = 0.$$

Since $\sin^2 x + \cos^2 x = 1$, this simplifies to

$$(\sin x - \cos x)(1 + 2\sin x\cos x) = 0.$$

The first factor gives $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

The second factor gives $2\sin x\cos x = -1$, so using $\sin(2x) = 2\sin x\cos x$, we have $\sin(2x) = -1$. Hence $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

Therefore, the sum of the four solutions is

$$\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4} = 4\pi.$$

You can also deduce the sum of the solutions by sketching the relevant graphs.

Question 7

Let $a > 0$. The curve $y = x^3 - 3a^2x$ and the line $y = -2a^3$ enclose a finite region. What is the area of this region?

A $\dfrac{27a^4}{4}$

B $\dfrac{9a^4}{2}$

C $\dfrac{27a^3}{4}$

D $\dfrac{81a^4}{8}$

Solution 7

Answer: A

The intersections satisfy $x^3 - 3a^2x = -2a^3$, so $x^3 - 3a^2x + 2a^3 = 0$.

Notice that $x = a$ is a root, so $(x - a)$ is a factor, and factorising eventually gives

$$x^3 - 3a^2x + 2a^3 = (x - a)^2(x + 2a).$$

Hence the curves intersect at $x = -2a$ and $x = a$.

Therefore the enclosed area is

$$\left| \int_{-2a}^{a} (x^3 - 3a^2x + 2a^3)\, dx \right|.$$

Evaluating,

$$\left[\frac{x^4}{4} - \frac{3a^2x^2}{2} + 2a^3x \right]_{-2a}^{a} = \frac{27a^4}{4}.$$

Remark: Using $\left|\int\right|$ is appropriate here, since the two curves do not cross again between the limits of integration. We therefore do not need to determine whether the line lies above or below the curve; taking the absolute value gives the bounded area directly.

Question 8

What is the sum of the x-coordinates of all real solutions (x, y) of the simultaneous equations $x^2 + xy - 2y^2 + x - y = 0$ and $2x^2 - 5xy + 2y^2 + 3x - 6y = 0$?

You are also given that the system have exactly 4 solutions.

A $-\frac{9}{2}$

B -5

C $-\frac{51}{10}$

D $-\frac{49}{10}$

Solution 8

Answer: D

Notice $x = y$ is a root of the first equation, hence $(x - y)$ is a factor, and the first equation factorises as $(x - y)(x + 2y + 1) = 0$.

Also notice $x = 2y$ is a root of the second equation, hence the second factorises as $(x-2y)(2x-y+3) = 0$.

We therefore solve the four possible pairs of linear equations.

From $x = y$ and $x = 2y$ we get $(0, 0)$.

From $x = y$ and $2x - y + 3 = 0$ we get $(-3, -3)$.

From $x + 2y + 1 = 0$ and $x = 2y$ we get $(-\frac{1}{2}, -\frac{1}{4})$.

From $x + 2y + 1 = 0$ and $2x - y + 3 = 0$ we get $(-\frac{7}{5}, \frac{1}{5})$.

These four solutions are distinct, by the given condition these are all the solutions. The sum of their x-coordinates is

$$0 - 3 - \frac{1}{2} - \frac{7}{5} = -\frac{49}{10}.$$

Remark: I did not actually need to give the condition that there are 4 solutions, since this can be deduced from the factorised forms of the equations. Indeed, the factorised forms are equivalent to the original equations, so any solution pair (x, y) must make one factor from each equation equal to zero. There are therefore only $2 \times 2 = 4$ possible pairs of linear equations to consider, and each pair can give at most one solution. Hence there can be at most 4 distinct solutions.

However, I thought the question was already challenging and long enough for a TMUA question without also asking you to make this deduction. :)

Question 9

In a non-degenerate triangle ABC, the side lengths opposite A, B, C are a, b, c respectively. Suppose $a^2 + ab - ac - 2b^2 + bc = 0$ and $c^2 = 3ab$. What is the value of C?

A $\dfrac{\pi}{3}$

B $\dfrac{\pi}{2}$

C $\dfrac{2\pi}{3}$

D $\dfrac{3\pi}{4}$

Solution 9

Answer: C

Notice $a = b$ is a root of $a^2 + ab - ac - 2b^2 + bc = 0$, therefore it factorises as $(a - b)(a + 2b - c) = 0$. Since the triangle is non-degenerate, $c < a + b$, so $a + 2b - c > b > 0$, and therefore in order for the equation to hold, $a = b$.

The second condition now gives $c^2 = 3a^2$.

By the cosine rule, $c^2 = a^2 + b^2 - 2ab \cos C$, substitute for c^2 and b, it becomes $3a^2 = 2a^2 - 2a^2 \cos C$. Hence $\cos C = -\dfrac{1}{2}$ and so $C = \dfrac{2\pi}{3}$.

Question 10

How many points (x, y) on the circle $2x^2+2y^2 = 5$ satisfy $2x^3-5x^2y-4x^2+xy^2+12xy+2y^3-8y^2 = 0$?

A 2

B 4

C 5

D 6

Solution 10

Answer: B

First notice that $x = y$ is a root of the cubic, so $(x - y)$ is a factor. Therefore the cubic can be factorised as

$$(x - y)(2x^2 - 3xy - 4x - 2y^2 + 8y) = 0.$$

Next notice that $x = 2y$ is a root of the quadratic factor, so $(x - 2y)$ is a factor. Hence the original cubic equation can be fully factorised as

$$(x - y)(x - 2y)(2x + y - 4) = 0.$$

Thus any point (x, y) satisfying both equations must lie on at least one of the three lines

$$x = y, \qquad x = 2y, \qquad 2x + y = 4.$$

The circle has centre the origin and radius $\sqrt{\frac{5}{2}}$.

The line $x = y$ passes through the origin, so it meets the circle twice.

The line $x = 2y$ also passes through the origin, so it meets the circle twice.

These two lines intersect only at the origin, which does not lie on the circle, so these four points are distinct.

For the final line $2x + y = 4$, there are two natural ways to show that it does not meet the circle.

One way is to find the perpendicular distance from the line to the origin. This can be shown to be $\frac{4}{\sqrt{5}}$, and since $\frac{16}{5} > \frac{5}{2}$, this distance is greater than the radius $\sqrt{\frac{5}{2}}$. Hence the line does not meet the circle.

Alternatively, substitute $y = 4 - 2x$ into the circle equation to get $2x^2 + 2(4-2x)^2 = 5$, which simplifies to $10x^2 - 32x + 27 = 0$. Its discriminant is $(-32)^2 - 4(10)(27) = -56 < 0$, so there are no real solutions. Hence the line does not meet the circle.

Hence there are 4 points.