TMUA TOPIC TEST

Cauchy-Schwarz and AM-GM Inequalities Topic Test Solutions

Time limit: 40 minutes

Calculators: not permitted

Format: 10 multiple-choice questions

This is a TMUA-style topic test: a timed set of TMUA-style questions, with a particular focus on a given topic. You can also sit this test online at jzmaths.com, where the result will count towards your mastery score.

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Question 1

For $x > 0$, what is the minimum value of $x + \frac{9}{x}$?

A 5

B 6

C 7

D 8

E $\dfrac{11}{2}$

Solution 1

Answer: B

Since $x > 0$, both x and $\frac{9}{x}$ are positive, so the AM-GM inequality gives

$$x + \frac{9}{x} \geq 2\sqrt{x \cdot \frac{9}{x}} = 2\sqrt{9} = 6.$$

Equality holds when $x = \frac{9}{x}$, which is clearly possible, so the minimum value is 6.

Remark: Of course, this question can also be solved by differentiation and finding the turning point, but the AM-GM inequality is simply faster here.

Question 2

For positive real numbers x and y, what is the smallest possible value of $(4x^2 + 9y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$?

A 13

B 25

C 30

D 36

E 49

Solution 2

Answer: B

Apply Cauchy-Schwarz to the pairs $(2x, 3y)$ and $(1/x, 1/y)$. This gives

$$(4x^2 + 9y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) \geq \left(2x\frac{1}{x} + 3y\frac{1}{y}\right)^2 = 25.$$

Equality can occur when the two pairs are proportional, which here is possible when $2x^2 = 3y^2$. Therefore the smallest possible value is 25.

Question 3

For $x > 1$, what is the smallest possible value of $\dfrac{x^2}{x-1}$?

A 4

B 3

C $\dfrac{9}{2}$

D 5

E $2+\sqrt{2}$

Solution 3

Answer: A

Let $t = x - 1$, so $t > 0$ and $x = t + 1$. Then

$$\frac{x^2}{x-1} = \frac{(t+1)^2}{t} = t + 2 + \frac{1}{t}.$$

The expression $t + \frac{1}{t}$ is one of the most typical applications of the AM-GM inequality. Since $t > 0$,

$$t + \frac{1}{t} \geq 2\sqrt{t \cdot \frac{1}{t}} = 2.$$

Hence

$$t + 2 + \frac{1}{t} \geq 4.$$

Equality occurs when $t = \frac{1}{t}$, which gives $t = 1$ and hence $x = 2$. Since $x > 1$, equality is attainable.

Therefore the smallest possible value is 4.

Question 4

Let x and y be positive real numbers with $2x + y = 8$. What is the maximum value of xy?

A 5

B 6

C 8

D 9

E 12

Solution 4

Answer: C

Applying the AM-GM inequality to $2x$ and y we get

$$\frac{2x+y}{2} \geq \sqrt{2x \cdot y}\ ,$$

this implies

$$\left(\frac{2x+y}{2}\right)^2 \geq 2x \cdot y\ .$$

Therefore

$$\left(\frac{8}{2}\right)^2 \geq 2x \cdot y \quad \Rightarrow \quad 8 \geq xy.$$

Equality holds when $2x = y$, together with $2x + y = 8$, this gives $x = 2$ and $y = 4$, so the maximum value is 8.

Question 5

Let a, b and c be positive real numbers. What is the minimum value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$?

A $\quad 1$

B $\quad \dfrac{3}{2}$

C $\quad 2$

D $\quad 6$

E $\quad 3$

Solution 5

Answer: E

Each of $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{c}{a}$ is positive, so the AM-GM inequality for three terms applies:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 3.$$

Equality holds when $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$, which is achieved when $a = b = c$. Hence equality is attainable, so the minimum value is 3.

Question 6

Let k be the smallest real number such that $(3x+4y)^2 \le k(x^2+2y^2)$ for all real numbers x and y. What is the value of k?

A 13

B 15

C 16

D 17

E 25

Solution 6

Answer: D

The inequality looks very similar in structure to the Cauchy-Schwarz inequality:

$$(aX+bY)^2 \le (a^2+b^2)(X^2+Y^2).$$

Indeed, let $X=x$, $Y=\sqrt{2}y$, $a=3$ and $b=2\sqrt{2}$. Then $x^2+2y^2=X^2+Y^2$, while $3x+4y=aX+bY$. So by Cauchy-Schwarz,

$$(3X+2\sqrt{2}Y)^2 \le (3^2+(2\sqrt{2})^2)(X^2+Y^2) = 17(X^2+Y^2).$$

Hence $k=17$ works.

Equality occurs when (X,Y) is proportional to $(a,b)=(3,2\sqrt{2})$. So just take $X=3$ and $Y=2\sqrt{2}$, which corresponds to $x=3$ and $y=2$. Then

$$(3x+4y)^2 = 17(x^2+2y^2).$$

The original inequality is required to be true for **all** real x and y for a given choice of k. If k were replaced by any number less than 17, say 16, then for $x=3$ and $y=2$ we would have

$$(3x+4y)^2 = 17(x^2+2y^2) > 16(x^2+2y^2),$$

so the inequality would fail for this pair. There are also other pairs of (x,y) for which it would fail as well, but one is sufficient.

Therefore no value of k less than 17 can work, and hence the smallest possible value is $k = 17$.

Question 7

For $-1 \leq x \leq 1$, what is the greatest possible value of $\sqrt{3x+3} + \sqrt{5-5x}$?

A $2\sqrt{2}$

B $\sqrt{10}$

C $\sqrt{15}$

D $\sqrt{17}$

E 4

Solution 7

Answer: E

Let $u = \sqrt{x+1}$ and $v = \sqrt{1-x}$. Then $u^2 + v^2 = 2$, and the expression becomes $\sqrt{3}u + \sqrt{5}v$. By Cauchy-Schwarz,

$$(\sqrt{3}u + \sqrt{5}v)^2 \leq (3+5)(u^2+v^2) = 16.$$

Since $\sqrt{3}u + \sqrt{5}v$ is non-negative, we may take the non-negative square root of both sides, giving

$$\sqrt{3}u + \sqrt{5}v \leq 4.$$

It remains to check that equality is possible. Equality in Cauchy-Schwarz occurs when $u : v = \sqrt{3} : \sqrt{5}$, so

$$\frac{u^2}{v^2} = \frac{3}{5}.$$

Since $u^2 = x+1$ and $v^2 = 1-x$,

$$\frac{x+1}{1-x} = \frac{3}{5}.$$

Hence $5(x+1) = 3(1-x)$, so $5x + 5 = 3 - 3x$, giving $8x = -2$ and therefore $x = -\frac{1}{4}$. This lies in the allowed interval $-1 \leq x \leq 1$, so equality is attainable.

Hence the greatest possible value is 4.

Question 8

A point $P = (x, y)$ lies on the line $5x + 12y = 30$. What is the smallest possible distance from P to the point $(1, 1)$?

A 1

B $\frac{4}{3}$

C $\sqrt{2}$

D $\frac{5}{4}$

E $\frac{4}{5}$

Solution 8

Answer: A

Let $X = x - 1$ and $Y = y - 1$, so that the point $(1, 1)$ is translated to the origin. Since $x = X + 1$ and $y = Y + 1$, the line $5x + 12y = 30$ becomes

$$5(X + 1) + 12(Y + 1) = 30 \quad \Leftrightarrow \quad 5X + 12Y = 13.$$

We can now equivalently find the minimum distance from the origin to the line $5X + 12Y = 13$. So we need to minimise the square of the distance, $X^2 + Y^2$. By Cauchy-Schwarz,

$$(5X + 12Y)^2 \leq (5^2 + 12^2)(X^2 + Y^2).$$

Since $5X + 12Y = 13$ and $5^2 + 12^2 = 169$,

$$169 \leq 169(X^2 + Y^2) \quad \Rightarrow \quad 1 \leq X^2 + Y^2.$$

Equality in Cauchy-Schwarz occurs when $X : Y = 5 : 12$. This is clearly possible, so equality is attainable.

Hence the smallest possible distance is $\sqrt{1} = 1$.

Remark: This question can be done without the initial translation, but I thought I would include it to demonstrate this useful technique, which can be relevant to other questions. Furthermore, Cauchy-Schwarz is not necessary here: you could instead find the normal to the line through the origin, find its intersection with the line, and then calculate the distance.

Question 9

Let $f(x) = ax + b$, where a and b are real numbers satisfying $a^2 + 4b^2 = 4$. Let $I = \int_0^2 f(x)\,dx$. What is the greatest possible value of I?

A $\sqrt{5}$

B 4

C 5

D $4\sqrt{5}$

E $2\sqrt{5}$

Solution 9

Answer: E

Evaluating the integral gives

$$I = \int_0^2 (ax + b)\,dx = 2a + 2b.$$

To maximize I, we clearly want a and b to be positive.

Let $c = 2b$. The condition becomes $a^2 + c^2 = 4$, while $I = 2a + c$. By Cauchy-Schwarz,

$$(2a + c)^2 \leq (2^2 + 1^2)(a^2 + c^2) = 5 \times 4 = 20.$$

Hence $I \leq 2\sqrt{5}$. Equality occurs when (a, c) is a positive multiple of $(2, 1)$ subject to $a^2 + c^2 = 4$, which clearly can be achieved. Therefore the greatest possible value of I is $2\sqrt{5}$.

Question 10

The quadratic equation $x^2 - 10x + c = 0$ has two positive real roots a and b, allowing $a = b$. What is the greatest possible value of $\frac{ab}{(a+1)(b+1)}$?

A $\dfrac{5}{7}$

B $\dfrac{5}{6}$

C $\dfrac{25}{36}$

D $\dfrac{25}{121}$

E $\dfrac{10}{21}$

Solution 10

Answer: C

By comparing coefficients, the roots satisfy $a + b = 10$ and $ab = c$. Since $a, b > 0$, AM-GM gives

$$\frac{a+b}{2} \geq \sqrt{ab},$$

so $5 \geq \sqrt{c}$ and hence $c \leq 25$, with equality when $a = b = 5$. Also,

$$\frac{ab}{(a+1)(b+1)} = \frac{c}{c+a+b+1} = \frac{c}{c+11}.$$

For positive c, this expression increases as c increases, since it can be written as $1 - \frac{11}{c+11}$. Therefore its greatest value occurs at $c = 25$ and is

$$\frac{25}{25+11} = \frac{25}{36}.$$