JZ Mock Set D Paper 1 Solutions

Recommended time: 2 hours

Calculators: not permitted

Format: 20 multiple-choice questions

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

Paper Guidance

- These papers contain only hard, very hard, and occasionally *unreasonably hard* TMUA-style questions :). I have deliberately left out the routine and easier material, so please do not be discouraged if you find a paper difficult or need longer than the recommended time. That is exactly how it is meant to feel. Take your time, enjoy the challenge, and remember that struggling with difficult problems is one of the best ways to get better.
- As a rough and deliberately conservative guide, a student who scores above 27 out of 40 across one full set, meaning both papers in four hours, is very likely already working at around the 7.0+ level on the current TMUA scale. However, this is only intended as a guide, not something to worry about too much. The main purpose of these papers is improvement rather than measurement: meet difficult problems, struggle with them, learn from them, and focus on getting better each time.

Best of luck, and enjoy the challenge!

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Question 1

Tags: Polynomial Expansions · Difficulty: 5.5

Find the coefficient of x^4 in the expansion of

$$\left(2 - x + \frac{x^2}{2}\right)^5.$$

A 50

B 60

C 70

D 80

E 90

Solution 1

Answer: E

One of a few standard questions. I may decide to replace it with a **non-standard** question at a later date, so take it easy for now!

Using the binomial expansion,

$$\left((2-x) + \frac{x^2}{2}\right)^5 = (2-x)^5 + 5(2-x)^4\left(\frac{x^2}{2}\right) + 10(2-x)^3\left(\frac{x^2}{2}\right)^2 + \cdots.$$

Only these three terms can contribute to the coefficient of x^4. Their contributions are

$$\begin{gathered}(2-x)^5 : \binom{5}{4}2(-x)^4 = 10x^4,\\ 5(2-x)^4\left(\frac{x^2}{2}\right) : 5\binom{4}{2}2^2(-x)^2\left(\frac{x^2}{2}\right) = 60x^4,\\ 10(2-x)^3\left(\frac{x^2}{2}\right)^2 : 10\cdot 2^3\left(\frac{x^4}{4}\right) = 20x^4.\end{gathered}$$

Therefore, the coefficient of x^4 is

$$10 + 60 + 20 = 90.$$

Question 2

Tags: General Trigonometry, General Number of Solutions · Difficulty: 6

How many solutions are there to the equation

$$|\tan x| + \frac{1}{\tan x} = 3 \quad \text{for} \quad 0^\circ \leq x \leq 360^\circ?$$

A 2

B 4

C 5

D 6

E 8

Solution 2

Answer: D

Let $t = \tan x$, where $t \neq 0$. The equation becomes

$$|t| + \frac{1}{t} = 3.$$

If $t > 0$, then $t + \frac{1}{t} = 3$, so $t^2 - 3t + 1 = 0$. This quadratic has two positive roots, you can see this by sketching suitable graphs, such as $y = 1$ and $y = 3t - t^2$, there is no need to solve the equation for t. Each root gives two values of x for $0^\circ \leq x \leq 360^\circ$. Hence there are 4 solutions.

If $t < 0$, then $-t + \frac{1}{t} = 3$, so $t^2 + 3t - 1 = 0$. This quadratic has one negative root, which gives two values of x in the given interval. Hence there are 2 more solutions.

Therefore, the total number of solutions is $4 + 2 = 6$.

Question 3

Tags: General Functions · Difficulty: 6

The functions f and g are defined for all real x by

$$f(x) = |x-2| + |x+2|$$

and

$$g(x) = x^2 - 2x + 4.$$

The function h is defined by

$$h(x) = g(f(x)).$$

What is the minimum value of $h(x)$?

A 10

B 11

C 12

D 13

E 14

F 15

Solution 3

Answer: C

Since $f(x) = |x-2| + |x+2|$ is the sum of the distances from x to 2 and -2, we have $f(x) \geq 4$, with equality for $-2 \leq x \leq 2$.

Also,

$$g(t) = t^2 - 2t + 4 = (t-1)^2 + 3.$$

For $t \geq 4$, $g(t)$ is increasing. Therefore, $g(f(x))$ is minimised when $f(x) = 4$.

Hence the minimum value is $g(4) = 16 - 8 + 4 = 12$.

Question 4

Tags: Differentiation · Difficulty: 6

Find the total length of interval(s) on which

$$f(x) = \frac{2}{5}x^{5/2} - 2x^{3/2} + 4x^{1/2} \qquad x > 0$$

is decreasing.

A $\frac{1}{2}$

B 1

C 2

D 3

E 4

F $\frac{5}{2}$

Solution 4

Answer: B

Differentiate:

$$f'(x) = x^{3/2} - 3x^{1/2} + 2x^{-1/2}.$$

Factorise:

$$f'(x) = x^{-1/2}(x^2 - 3x + 2) = x^{-1/2}(x-1)(x-2).$$

Since $x > 0$, we have $x^{-1/2} > 0$, so the sign of $f'(x)$ is the sign of $(x-1)(x-2)$.

The function is decreasing when $f'(x) \leq 0$, so

$$(x-1)(x-2) \leq 0.$$

Hence $1 \leq x \leq 2$. Therefore the total length of the interval on which f is decreasing is $2 - 1 = 1$.

Question 5

Tags: Graphs of Functions, Inequalities · Difficulty: 6

Find the area of the region in the (x, y)-plane consisting of all points satisfying both

$$|x| + |y| \leq 2 \quad \text{and} \quad |x-1| + |y| \leq 2.$$

A 3

B $\frac{9}{4}$

C $\frac{7}{2}$

D 4

E $\frac{9}{2}$

F 6

G $\frac{15}{2}$

H 8

Solution 5

Answer: E

The inequality $|x| + |y| \leq 2$ describes a diamond centred at $(0, 0)$, while $|x-1| + |y| \leq 2$ describes an identical diamond centred at $(1, 0)$.

Sketch out the two diamonds, then you will see that their common region is also a diamond. Its leftmost and rightmost vertices are $(-1, 0)$ and $(2, 0)$, so its horizontal diagonal has length 3.

Its upper and lower vertices are

$$\left(\frac{1}{2}, \frac{3}{2}\right) \quad \text{and} \quad \left(\frac{1}{2}, -\frac{3}{2}\right),$$

so its vertical diagonal also has length 3.

Therefore, the area of the common region is

$$\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2}.$$

Question 6

Tags: Sequences and Series · Difficulty: 6

An arithmetic progression has $(2n + 1)$ terms, where n is a positive integer. Let S_o denote the sum of the terms in odd-numbered positions (the 1st, 3rd, 5th, ... terms) and let S_e denote the sum of the terms in even-numbered positions (the 2nd, 4th, 6th, ... terms).

It is given that

$$S_o + S_e = 505 \qquad \text{and} \qquad S_o - S_e = 5.$$

What is the value of n?

A 40

B 50

C 51

D 55

E 56

Solution 6

Answer: B

Since there are an odd number $2n + 1$ terms, there must be a middle term, call it M, then $S_o = (n + 1)M$ and $S_e = nM$. Therefore

$$\begin{aligned} S_o + S_e &= (n + 1)M + nM = (2n + 1)M, \\ S_o - S_e &= (n + 1)M - nM = M. \end{aligned}$$

From $S_o - S_e = 5$ we get $M = 5$. Substituting into $S_o + S_e = 505$ gives $(2n + 1) \cdot 5 = 505$, so $2n + 1 = 101$ and hence $n = 50$.

Question 7

Tags: General Algebra, Sequences and Series · Difficulty: 6

Find the value of

$$\sum_{r=1}^{100} \frac{1}{\sqrt{r}+\sqrt{r-1}}.$$

A 0

B 1

C 9

D 10

E 11

F 20

Solution 7

Answer: D

Rationalising each term gives

$$\frac{1}{\sqrt{r}+\sqrt{r-1}} = \frac{1}{\sqrt{r}+\sqrt{r-1}} \times \frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r}-\sqrt{r-1}} = \sqrt{r}-\sqrt{r-1}.$$

Therefore,

$$\begin{aligned}\sum_{r=1}^{100} \frac{1}{\sqrt{r}+\sqrt{r-1}} &= \sum_{r=1}^{100} \left(\sqrt{r}-\sqrt{r-1}\right) \\ &= \sqrt{100}-\sqrt{0} \\ &= 10.\end{aligned}$$

Question 8

Tags: Circle Geometry, General Algebra · Difficulty: 6.5

The two curves C_1 and C_2 in the xy-plane have equations

$$x^2 + y^2 - 2y - 12 = 0 \text{ and } x^2 + y^2 - 2x - 4y = 0.$$

Let P be a point on C_1 and Q be a point on C_2. What is the **least** possible value of the length of PQ?

A $\sqrt{2} + \sqrt{5} - \sqrt{13}$

B $\sqrt{13} - \sqrt{5} - \sqrt{2}$

C $\sqrt{13} - \sqrt{2}$

D 0

E $\sqrt{13} + \sqrt{5} - \sqrt{2}$

Solution 8

Answer: D

C_1: $x^2 + (y-1)^2 = 13$, so its centre is $A = (0, 1)$ and its radius is $\sqrt{13}$.

C_2: $(x-1)^2 + (y-2)^2 = 5$, so its centre is $B = (1, 2)$ and its radius is $\sqrt{5}$.

Now $AB = \sqrt{2}$. The circles intersect if $\sqrt{2} + \sqrt{5} > \sqrt{13}$. Since

$$(\sqrt{2} + \sqrt{5})^2 = 7 + 2\sqrt{10} > 13,$$

the circles intersect.

Hence we may choose $P = Q$ at an intersection point, so the least possible value of PQ is 0. Indeed the whole point of this question is to get you to determine how close they really are.

Alternatively, shortcut to mushrooms: subtracting the two equations gives $2x + 2y - 12 = 0$, so $y = 6 - x$. Substituting this into either equation gives

$$x^2 - 5x + 6 = 0,$$

which has two distinct real roots. Hence the circles intersect, so the least possible value of PQ is 0.

Question 9

Tags: General Trigonometry, General Algebra · Difficulty: 6.5

Let x be any real number, find the minimum value of

$$4^{\sin^2 x} - 5 \cdot 2^{\sin^2 x} + 7.$$

A There is no minimum value.

B 1

C $\dfrac{5}{4}$

D 3

E 7

F $\dfrac{3}{4}$

Solution 9

Answer: B

Another completing the square question with a twist!

Let $u = 2^{\sin^2 x}$. Since $\sin^2 x$ takes every value in $[0, 1]$ (and only those values) as x varies over $\mathbb{R}$, u ranges over $[2^0, 2^1] = [1, 2]$.

Note that $4^{\sin^2 x} = (2^{\sin^2 x})^2 = u^2$, so the expression becomes

$$g(u) = u^2 - 5u + 7 = \left(u - \frac{5}{2}\right)^2 + \frac{3}{4}.$$

The minimum of g is $\frac{3}{4}$ if u is allowed to be $\frac{5}{2}$, it is not! As u varies between 1 to 2, given the shape of $g(u)$, the minimum is clearly at $u = 2$:

$$g(2) = 4 - 10 + 7 = 1.$$

Question 10

Tags: Remainder Theorem, Polynomial Expansions · Difficulty: 6.5

Given that $x^2 - 4x + 1$ is a factor of

$$2x^4 - 11x^3 + bx^2 + cx + 6,$$

what is the value of $b + c$?

A -47

B -27

C -7

D 13

E 47

F 53

Solution 10

Answer: C

Since $x^2 - 4x + 1$ is a factor and the quartic has leading coefficient 2 and constant term 6, write

$$2x^4 - 11x^3 + bx^2 + cx + 6 = (x^2 - 4x + 1)(2x^2 + px + q).$$

Expanding the right-hand side gives

$$2x^4 + (p-8)x^3 + (q - 4p + 2)x^2 + (p - 4q)x + q.$$

Comparing coefficients:

$$\begin{aligned} x^0 &: \quad q = 6, \\ x^3 &: \quad p - 8 = -11 \implies p = -3, \\ x^2 &: \quad b = q - 4p + 2 = 6 + 12 + 2 = 20, \\ x^1 &: \quad c = p - 4q = -3 - 24 = -27. \end{aligned}$$

Therefore $b + c = 20 + (-27) = -7$.

Remark: The roots of $x^2 - 4x + 1$ are $x = 2 \pm \sqrt{3}$, so direct substitution would not be wise!

Question 11

Tags: General Number of Solutions, Graphs of Functions · Difficulty: 7

Let $f(x) = -x^3 + kx^2 + 3$, where k is a real constant. Suppose that $f(x) = 0$ has exactly 3 distinct real roots.

Which of the following equations must also have exactly 3 distinct real roots?

I $\quad -f(x) + a = 0$, where a is an arbitrary real constant

II $\quad 5 - 2f(2 - 3x) = 0$

III $\quad f(-|x|) - 3 = 0$

A I only

B II only

C III only

D I and III only

E I and II only

F all of them

G II and III only

H none of them

Solution 11

Answer: G

We have $f'(x) = x(2k - 3x)$, so the stationary points are at $x = 0$ and $x = \frac{2k}{3}$.

Since $f(0) = 3 > 0$ and $f(x) = 0$ has three distinct real roots, $x = 0$ must be the local maximum and the other stationary point must be a local minimum below the x-axis. Hence $\frac{2k}{3} < 0$, so $k < 0$.

Equation I need not have three distinct real roots, since a can be chosen sufficiently large that both stationary values of $-f(x) + a$ are positive, giving only one real root.

For II, let $u = 2 - 3x$. Then

$$5 - 2f(2-3x) = 0 \quad \Longleftrightarrow \quad f(u) = \frac{5}{2}.$$

The local maximum value is 3 and the local minimum value is negative. Since $0 < \frac{5}{2} < 3$, the line $y = \frac{5}{2}$ intersects the graph three times. As $u = 2 - 3x$ is one-to-one, II has three distinct real roots.

For III,

$$f(-|x|) - 3 = |x|^3 + kx^2 = x^2(|x| + k).$$

Since $k < 0$, the equation has the three distinct roots $x = 0$, $x = k$ and $x = -k$.

Therefore, II and III only must have exactly three distinct real roots.

Question 12

Tags: Transformation of Graphs · Difficulty: 7

The curve $x = y^2 - 8y + 19$ is rotated $90°$ clockwise about $P_1 = (1, 2)$, and then $90°$ clockwise about P_2. The final curve is $x = -y^2 + 2y - 7$. Given P_2 lies on the line $y = 5$, find the x-coordinate of P_2.

A -1

B -2

C -3

D 0

E 1

F 2

G 3

H 4

Solution 12

Answer: A

Let $P_2 = (a, 5)$.

First rewrite the original curve:

$$x = y^2 - 8y + 19 = (y - 4)^2 + 3.$$

So its vertex is $(3, 4)$.

The final curve is

$$x = -y^2 + 2y - 7 = -(y - 1)^2 - 6.$$

So its vertex is $(-6, 1)$.

A rotation maps the vertex of the parabola to the vertex of the new parabola, so track the vertex only.

The key observation here is: a 90° clockwise rotation about (h, k) sends a point (x, y) to $(h + (y - k), k - (x - h)) = (h + y - k, k - x + h)$, which can be seen from a diagram (check it out yourself).

Rotating $(3, 4)$ about $P_1 = (1, 2)$ gives

$$(1 + 4 - 2, 2 - 3 + 1) = (3, 0).$$

Now rotate $(3, 0)$ about $P_2 = (a, 5)$:

$$(a + 0 - 5, 5 - 3 + a) = (a - 5, a + 2).$$

This must equal the final vertex $(-6, 1)$, so

$$a - 5 = -6$$

and hence $a = -1$.

Therefore the x-coordinate of P_2 is -1.

Question 13

Tags: Transformation of Graphs, General Trigonometry · Difficulty: 7

The graph of $y = \sin x$ is transformed by the following five operations, applied in the order given:

I a reflection in the line $y = -3$;

II a stretch by scale factor 2 parallel to the x-axis;

III a translation of $\frac{\pi}{3}$ in the negative x-direction.

IV a reflection in the line $x = 0$;

V a reflection in the line $x = \frac{4\pi}{3}$.

Which of the following is the equation of the resulting curve?

A

$$y = -3 + \sin\left(\frac{x}{2} + \frac{5\pi}{12}\right)$$

B

$$y = -6 - \sin\left(\frac{x}{2} - \frac{5\pi}{12}\right)$$

C

$$y = -6 + \sin\left(\frac{x}{2} - \frac{\pi}{6}\right)$$

D

$$y = -3 + \sin\left(\frac{x}{2} - \frac{5\pi}{12}\right)$$

E

$$y = 6 + \sin\left(\frac{x}{2} - \frac{5\pi}{12}\right)$$

Solution 13

Answer: C

The reflection in the line $y = -3$ affects only the y-coordinate, while transformations II to V affect only the x-coordinate. We may therefore apply transformation I last.

Starting with $y = \sin x$, the stretch and translation give

$$y = \sin\left(\frac{x}{2} + \frac{\pi}{6}\right).$$

Reflecting in $x = 0$ gives
$$y = \sin\left(-\frac{x}{2} + \frac{\pi}{6}\right).$$
The lines of symmetry of this curve satisfy
$$-\frac{x}{2} + \frac{\pi}{6} = \frac{\pi}{2} + n\pi,$$
so $x = -\frac{2\pi}{3} - 2n\pi$. The least positive value is $\frac{4\pi}{3}$, so reflection in $x = \frac{4\pi}{3}$ leaves the curve unchanged.

Finally, reflection in $y = -3$ changes y to $-6 - y$, giving
$$y = -6 - \sin\left(-\frac{x}{2} + \frac{\pi}{6}\right) = -6 + \sin\left(\frac{x}{2} - \frac{\pi}{6}\right).$$

Question 14

Tags: Sequences and Series, Trig Equation Number of Solutions · Difficulty: 7

The equation

$$\cos x + \cos^3 x + \cos^5 x + \cos^7 x + \ldots = \tan x$$

has how many solutions for $0 \le x < 2\pi$?

A 0

B 1

C 2

D 3

E 4

F infinitely many

Solution 14

Answer: C

The left-hand side is a geometric series with first term $\cos x$ and common ratio $\cos^2 x$. It converges for $\cos^2 x < 1$, so we must exclude $x = 0$ and $x = \pi$, where $\cos^2 x = 1$. The right-hand side $\tan x$ is not defined for $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, so we also exclude these x.

When the series converges, its sum to infinity is

$$\frac{\cos x}{1 - \cos^2 x} = \frac{\cos x}{\sin^2 x}.$$

Substituting into the equation, we get

$$\frac{\cos x}{\sin^2 x} = \tan x \quad \iff \quad \frac{\cos x}{\sin^2 x} = \frac{\sin x}{\cos x}.$$

Multiplying both sides by $\sin^2 x \cdot \cos x$, which is nonzero as we precisely excluded the values of x for which they are zero, simplify, eventually gives

$$\sin^3 x + \sin^2 x - 1 = 0$$

Let $u = \sin x$. Since $-1 \leq u \leq 1$, we need to count the solutions of $u^3 = 1 - u^2$ in this interval. For $-1 \leq u \leq 0$, we have $u^3 \leq 0$ and $1 - u^2 \geq 0$, so there are no solutions. For $0 \leq u \leq 1$, the function u^3 is strictly increasing from 0 to 1, while $1 - u^2$ is strictly decreasing from 1 to 0. Hence they intersect exactly once, at some $u = k$ with $0 < k < 1$. The equation $\sin x = k$ has exactly two solutions for $0 \leq x < 2\pi$, so the original equation has 2 solutions.

Question 15

Tags: Circle Geometry, Geometry · Difficulty: 7.5

The circle C is inscribed in the triangle bounded by the three lines

$$y = 0, \qquad y = \sqrt{3}x, \qquad x + y = \sqrt{1} + \sqrt{2} + \sqrt{3}.$$

You may use find $\tan 22.5^\circ = \sqrt{2} - 1$ useful.

Which of the following is the equation of C?

A

$$(x - \sqrt{3})^2 + (y - 1)^2 = 1$$

B

$$(x - 1)^2 + (y - \sqrt{3})^2 = 1$$

C

$$(x - \sqrt{3})^2 + (y - \sqrt{2})^2 = 1$$

D

$$(x - \sqrt{2})^2 + (y - 1)^2 = 2$$

E

$$(x - \sqrt{2})^2 + (y - \sqrt{2})^2 = 2$$

Solution 15

Answer: A

The centre of an inscribed circle is the intersection of the internal angle bisectors. (GCSEish knowledge!)

At the origin, the angle between $y = 0$ and $y = \sqrt{3}x$ is 60°, so the angle bisector makes an angle of 30° with the positive x-axis. Therefore its equation is

$$y = x \tan 30^\circ = \frac{x}{\sqrt{3}}.$$

The line $x + y = 1 + \sqrt{2} + \sqrt{3}$ meets the x-axis at

$$(1 + \sqrt{2} + \sqrt{3}, 0).$$

At this point, the internal angle is between the negative x-axis and the line $x + y = 1 + \sqrt{2} + \sqrt{3}$. Since the line has gradient -1, the angle bisector makes an angle of 22.5° above the negative x-axis.

Using $\tan 22.5^\circ = \sqrt{2} - 1$, the second angle bisector is

$$y = (\sqrt{2} - 1)(1 + \sqrt{2} + \sqrt{3} - x).$$

Now intersect the two angle bisectors:

$$\frac{x}{\sqrt{3}} = (\sqrt{2} - 1)(1 + \sqrt{2} + \sqrt{3} - x).$$

Try $x = \sqrt{3}, \sqrt{2}, 1$, from the options, we find that $x = \sqrt{3}$. Then the left-hand side is 1, and the right-hand side is $(\sqrt{2} - 1)(1 + \sqrt{2}) = 1$.

So $x = \sqrt{3}$, and hence $y = 1$.

Therefore the centre is $(\sqrt{3}, 1)$.

Since the circle touches the line $y = 0$, its radius is 1. Therefore the equation of the circle is

$$(x - \sqrt{3})^2 + (y - 1)^2 = 1.$$

Question 16

Tags: Integration, Sequences and Series · Difficulty: 7.5

Let f be a real-valued function such that

$$\int_0^1 f(x)\,dx = 1$$

and $f(x) = f(x+1)$ for all real x.

Find

$$\sum_{r=1}^{100} \int_0^r f(r^r x)\,dx.$$

A 50

B 100

C 101

D 5050

E 2050

F 4950

Solution 16

Answer: D

Remark: The functions $f(x), f(4x), f(27x), f(256x), \ldots$ do not look especially friendly. You may wonder how much the coefficients of x inside f matter. It turns out that they do not matter at all!

For any positive integer k, the graph of $y = f(kx)$ is obtained from the graph of $y = f(x)$ by a horizontal stretch with scale factor $\frac{1}{k}$.

Since f has period 1, the function $f(kx)$ completes k periods on the interval $[0, 1]$. Each compressed period has integral

$$\frac{1}{k}\int_0^1 f(x)\,dx = \frac{1}{k}.$$

Therefore,

$$\int_0^1 f(kx)\,dx = k \cdot \frac{1}{k} = 1.$$

For each r, the number r^r is a positive integer, so

$$\int_0^1 f(r^r x)\,dx = 1.$$

Also, since r^r is an integer,

$$f(r^r(x+1)) = f(r^r x + r^r) = f(r^r x),$$

so $f(r^r x)$ has period 1.

Hence the interval $[0, r]$ consists of r intervals of length 1, each contributing an integral of 1. Therefore,

$$\int_0^r f(r^r x)\,dx = r.$$

Thus the required sum is

$$\sum_{r=1}^{100} r = 1 + 2 + 3 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050.$$

Question 17

Tags: General Algebra, Polynomial Expansions · Difficulty: 8

A real number x satisfies

$$x^2 - 3x + 1 = 0.$$

For each non-negative integer n, define

$$p_n = x^n + \frac{1}{x^n}.$$

Find the value of p_6.

You may find it helpful to consider a recurrence relation of the form $p_n = 3p_{n-1} + kp_{n-2}$ for some constant k.

A 322

B 123

C 312

D 151

E 276

F 258

Solution 17

Answer: A

Remark: If a question says, **You may find it helpful**, this is often a euphemism for, **If you do not use this, you are unlikely to get very far!** So, although it may sound optional, you would be wise to follow the hint.

Since $x^2 - 3x + 1 = 0$ and $x \neq 0$, dividing by x gives

$$x + \frac{1}{x} = 3.$$

For $n \geq 2$,

$$\begin{aligned} 3p_{n-1} &= \left(x + \frac{1}{x}\right)\left(x^{n-1} + \frac{1}{x^{n-1}}\right) \\ &= x^n + x^{n-2} + \frac{1}{x^{n-2}} + \frac{1}{x^n} \\ &= p_n + p_{n-2}. \end{aligned}$$

Therefore,

$$p_n = 3p_{n-1} - p_{n-2},$$

so $k = -1$.

The initial values are $p_0 = 2$ and $p_1 = x + \frac{1}{x} = 3$.

We can now apply the recurrence repeatedly:

$$\begin{aligned} p_2 &= 3p_1 - p_0 = 3(3) - 2 = 7, \\ p_3 &= 3p_2 - p_1 = 3(7) - 3 = 18, \\ p_4 &= 3p_3 - p_2 = 3(18) - 7 = 47, \\ p_5 &= 3p_4 - p_3 = 3(47) - 18 = 123, \\ p_6 &= 3p_5 - p_4 = 3(123) - 47 = 322. \end{aligned}$$

Hence

$$p_6 = 322.$$

Question 18

Tags: Exponentials and Logarithms, Integration · Difficulty: 8

Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x.

Find the value of

$$\int_{100}^{1} \log_{10}\left(\frac{\lfloor 2x \rfloor}{\lfloor 2x \rfloor + 1}\right) dx.$$

A -1

B $\log_{10} 99$

C $\log_{10} 101$

D $-\log_{10} 99$

E $-\log_{10} 101$

F 1

Solution 18

Answer: F

Remark: Students often find integrals involving step functions difficult, so here is an opportunity to practise! I have added a couple of extra twists to this one. Do you like them?!

Reversing the limits changes the sign of the integral:

$$\int_{100}^{1} \log_{10}\left(\frac{\lfloor 2x \rfloor}{\lfloor 2x \rfloor + 1}\right) dx = -\int_{1}^{100} \log_{10}\left(\frac{\lfloor 2x \rfloor}{\lfloor 2x \rfloor + 1}\right) dx.$$

For $\frac{k}{2} \le x < \frac{k+1}{2}$, we have $\lfloor 2x \rfloor = k$. From $x = 1$ to $x = 100$, the values of $\lfloor 2x \rfloor$ are $2, 3, \ldots, 199$, each lasting for an interval of length $\frac{1}{2}$. Therefore

$$\int_{1}^{100} \log_{10}\left(\frac{\lfloor 2x \rfloor}{\lfloor 2x \rfloor + 1}\right) dx = \frac{1}{2}\sum_{k=2}^{199} \log_{10}\left(\frac{k}{k+1}\right).$$

So

$$= \frac{1}{2}\log_{10}\left(\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdots\frac{199}{200}\right) = \frac{1}{2}\log_{10}\left(\frac{1}{100}\right) = -1.$$

Hence

$$\int_{100}^{1} \log_{10}\left(\frac{\lfloor 2x \rfloor}{\lfloor 2x \rfloor + 1}\right) dx = -(-1) = 1.$$

So the answer is **1**, how neat!

Question 19

Tags: Integration, General Functions · Difficulty: 8

Let functions $f_0(x) = |x|$ and $f_{n+1}(x) = |1 - f_n(x)|$, find

$$\int_0^{100} f_{100}(x)\,dx.$$

A 5000

B 2500

C 100

D 50

E 25

Solution 19

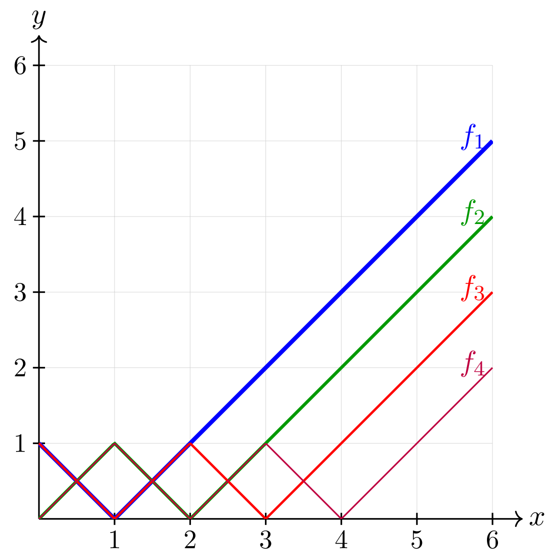
Answer: D

This lovely question occurred to me while I was watching R2Drew2's video on my first mock paper. In one question, he drew some very nice sketches of the sine and cosine graphs, and somehow this idea came into my head. So here it is - blame him!

The trick is to sketch successive functions $f_r(x)$ and look for a pattern. Since $|X| = |-X|$, it is easier to write the recurrence relation as

$$f_{n+1}(x) = |f_n(x) - 1|.$$

This corresponds to translating the graph down by 1 unit and then applying the modulus transformation. As you sketch the functions, you will see what is happening:



Therefore, the area described by the integral is

$$\int_0^{100} f_{100}(x)\,dx = 100 \cdot \frac{1}{2} = 50.$$

Question 20

Tags: Ratio and Proportion, General Algebra · Difficulty: 8.5

The positive real numbers a, b, c, d form a geometric progression in this order. Given that

$$a + d = 35 \qquad \text{and} \qquad b + c = 30,$$

find the value of

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

A $\dfrac{65}{216}$

B $\dfrac{13}{7}$

C $\dfrac{13}{6}$

D $\dfrac{1}{216}$

E $\dfrac{5}{216}$

F $\dfrac{13}{210}$

Solution 20

Answer: A

Remark: This is an example of a challenging algebra problem. In problems of this kind, it is important to look for opportunities to simplify the algebra as early as possible.

Step 1: A useful identity for a 4-term geometric progression. Write $b = ar$, $c = ar^2$, $d = ar^3$ for some common ratio $r > 0$. Then

$$ad = a \cdot ar^3 = a^2r^3 \qquad \text{and} \qquad bc = ar \cdot ar^2 = a^2r^3,$$

so $ad = bc$. Calling this common product P, we have

$$\frac{1}{a} + \frac{1}{d} = \frac{a+d}{ad} = \frac{a+d}{P}, \qquad \frac{1}{b} + \frac{1}{c} = \frac{b+c}{bc} = \frac{b+c}{P}.$$

Hence
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{(a+d)+(b+c)}{P}=\frac{65}{P}.$$

Step 2: Find $P = ad$. Using the geometric progression form,

$$a+d=a(1+r^3)=a(1+r)(1-r+r^2), \qquad b+c=ar(1+r).$$

Dividing,
$$\frac{a+d}{b+c}=\frac{1-r+r^2}{r}=\frac{35}{30}=\frac{7}{6}.$$

So $6(1-r+r^2)=7r$, i.e. $6r^2-13r+6=0$, giving $r=\frac{3}{2}$ or $r=\frac{2}{3}$.

Either choice gives the same value of $P = ad$. For $r = 3/2$: $a(1+r)(1-r+r^2)=a\cdot\frac{5}{2}\cdot\frac{7}{4}=\frac{35a}{8}=35$, so $a = 8$. Then $d = ar^3 = 8\cdot\frac{27}{8}=27$, and $P = ad = 216$. For $r = 2/3$ the values of a and d swap, and P is unchanged.

Step 3: Combine.
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{65}{216}.$$