JZ Mock Set C Paper 2

Solutions

Time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

Average difficulty: 7.3

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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Question 1

Tags: Inequalities · Difficulty: 5.5

Let a, b, c be real numbers with $a \geq b$. Which of the following statements **must** be true?

1. $a^3 \geq b^3$
2. $a^2 \geq b^2$
3. $ac \geq bc$
4. $ac^2 \geq bc^2$

A none of them

B 1 only

C 4 only

D 1 and 2 only

E 1 and 3 only

F 1 and 4 only

G 2 and 4 only

H 1, 2 and 4 only

I 1, 3 and 4 only

J 1, 2, 3 and 4

Solution 1

Answer: F

Examine each statement under the hypothesis $a \geq b$ (with $a, b, c \in \mathbb{R}$, no further restrictions).

Statement 1: $a^3 \geq b^3$. The function $x \mapsto x^3$ is strictly increasing on all of $\mathbb{R}$, so $a \geq b \Rightarrow a^3 \geq b^3$. **True**.

Statement 2: $a^2 \geq b^2$. Squaring is **not** monotonic on $\mathbb{R}$. Counterexample: $a = 1$, $b = -2$. Then $a \geq b$ but $a^2 = 1 < 4 = b^2$. **False**.

Statement 3: $ac \geq bc$. Multiplying an inequality by c preserves its direction only when $c \geq 0$. Counterexample: $a = 1$, $b = 0$, $c = -1$. Then $ac = -1 < 0 = bc$. **False**.

Statement 4: $ac^2 \geq bc^2$. Here the multiplier is $c^2 \geq 0$ regardless of the sign of c. Since $a - b \geq 0$ and $c^2 \geq 0$, we have $(a - b)c^2 \geq 0$, i.e. $ac^2 \geq bc^2$. **True**.

Only statements 1 and 4 must hold, so the answer is **F**.

Question 2

Tags: General Trigonometry · Difficulty: 6

What is the value, in radians, of the second-largest angle x in the range $0 \leq x \leq 2\pi$ that satisfies the equation

$$\tan^2(2x)\cos^2(2x) + 5\cos^2(2x) = 4?$$

A $\frac{\pi}{12}$

B $\frac{5\pi}{12}$

C $\frac{7\pi}{6}$

D $\frac{13\pi}{12}$

E $\frac{19\pi}{12}$

F $\frac{23\pi}{12}$

G $\frac{25\pi}{12}$

Solution 2

Answer: E

Use $\tan(2x) = \frac{\sin(2x)}{\cos(2x)}$ to rewrite $\tan^2(2x)\cos^2(2x) = \sin^2(2x)$ (valid since the equation requires $\cos(2x) \neq 0$). The equation becomes

$$\sin^2(2x) + 5\cos^2(2x) = 4.$$

Applying $\sin^2(2x) + \cos^2(2x) = 1$ to substitute $\sin^2(2x) = 1 - \cos^2(2x)$ gives

$$1 - \cos^2(2x) + 5\cos^2(2x) = 4 \implies 4\cos^2(2x) = 3 \implies \cos(2x) = \pm\frac{\sqrt{3}}{2}.$$

Since $0 \leq x \leq 2\pi$, the argument $2x$ ranges over $[0, 4\pi]$. In this extended range, $\cos(2x) = \frac{\sqrt{3}}{2}$ gives $2x \in \{\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}\}$, and $\cos(2x) = -\frac{\sqrt{3}}{2}$ gives $2x \in \{\frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}\}$.

Dividing each by 2 yields the eight solutions

$$x \in \left\{ \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12} \right\}.$$

The largest is $\frac{23\pi}{12}$ and the second-largest is $\frac{19\pi}{12}$. The answer is **E**.

The question can be done faster by a graphic apprach, sketching out the graphs of $\cos(2x)$ against $y = \pm\frac{\sqrt{3}}{2}$.

Question 3

Tags: Logic Deduction · Difficulty: 6

Six sealed black bags are labelled P, Q, R, S, T and U. Each bag contains the same number of balls. This number is a non-zero whole number from 1 to 10 inclusive.

Each bag has a label with a statement written on it. The statement may be true or false.

Bag P has a label which says: "The number of balls in this bag is a multiple of 3."

Bag Q has a label which says: "The number of balls in this bag is even."

Bag R has a label which says: "The number of balls in this bag is prime."

Bag S has a label which says: "The number of balls in this bag is at most 4."

Bag T has a label which says: "The number of balls in this bag is at least 7."

Bag U has a label which says: "The number of balls in this bag is a perfect square."

Given that exactly one of the six statements is true, which bag has the true statement on its label?

A A: P

B B: Q

C C: R

D D: S

E E: T

F F: U

Solution 3

Answer: C

Let the common number of balls in each bag be n. Then n is one of

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10.$$

We check which labels would be true for each possible value of n.

n	True labels
1	$S,\ U$
2	$Q,\ R,\ S$
3	$P,\ R,\ S$
4	$Q,\ S,\ U$
5	R
6	$P,\ Q$
7	$R,\ T$
8	$Q,\ T$
9	$P,\ T,\ U$
10	$Q,\ T$

Only $n = 5$ gives exactly one true statement.

When $n = 5$, the only true label is R, because 5 is prime.

So the bag with the true statement on its label is

$$\mathbf{R}.$$

Question 4

Tags: Logic Deduction, Inequalities · Difficulty: 6.5

Let $x \neq 0$ and

$$f(x) = \frac{3}{7}x^{7/3} + \frac{3}{4}x^{4/3} - 6x^{1/3}.$$

A student attempts to find the values of x for which the curve is increasing.

Their working is shown below.

I. $f'(x) = x^{4/3} + x^{1/3} - 2x^{-2/3}$

II. $f'(x) = x^{-2/3}(x^2 + x - 2)$

III. $x^{-2/3}(x^2 + x - 2) \geq 0$

IV. $x^2 + x - 2 \geq 0$

V. $(x - 1)(x + 2) \geq 0$

VI. $x \leq -2$ or $x \geq 1$

Which of the following is true?

A The first error is in step **I**.

B The first error is in step **II**.

C The first error is in step **III**.

D The first error is in step **IV**.

E The first error is in step **V**.

F The first error is in step **VI**.

G The student's answer is correct.

Solution 4

Answer: G

For $x \neq 0$, differentiating term by term gives

$$f'(x) = \frac{3}{7} \cdot \frac{7}{3}x^{4/3} + \frac{3}{4} \cdot \frac{4}{3}x^{1/3} - 6 \cdot \frac{1}{3}x^{-2/3}$$

so

$$f'(x) = x^{4/3} + x^{1/3} - 2x^{-2/3}.$$

Therefore step I is correct.

Now

$$x^{4/3} = x^{-2/3}x^2 \quad \text{and} \quad x^{1/3} = x^{-2/3}x,$$

so

$$f'(x) = x^{-2/3}(x^2 + x - 2).$$

Therefore step II is correct.

The curve is increasing when $f'(x) \geq 0$, so step III is correct. Since $x \neq 0$, we have $x^{-2/3} > 0$, so dividing by $x^{-2/3}$ does not change the direction of the inequality. Hence

$$x^2 + x - 2 \geq 0.$$

Therefore step IV is correct.

Next,

$$x^2 + x - 2 = (x - 1)(x + 2),$$

so step V is correct. Finally, $(x - 1)(x + 2) \geq 0$ outside the two roots, giving

$$x \leq -2 \quad \text{or} \quad x \geq 1.$$

Therefore step VI is correct.

So the student's working is correct, and the correct option is **G**.

Question 5

Tags: Remainder Theorem · Difficulty: 6.5

A polynomial $p(x)$ satisfies $p(2) = 7$ and $p(-3) = -8$.

Which of the following can be deduced, where $q(x)$ denotes some polynomial?

A $p(x) = (x-2)(x+3)\,q(x) + 3x + 1$

B $p(x) = (x-2)(x+3)\,q(x) - 3x + 1$

C $p(x) = (x-2)(x+3)\,q(x) + 3x - 1$

D $p(x) = (x+2)(x-3)\,q(x) + 3x + 1$

E $p(x) = (x-2)(x+3)\,q(x) - x + 5$

F $p(x) = (x-2)(x+3)\,q(x) + x - 5$

G $p(x) = (x+2)(x-3)\,q(x) - 3x - 1$

H $p(x) = (x-2)(x+3)\,q(x) + \frac{15}{2}x - \frac{1}{2}$

Solution 5

Answer: A

Divide $p(x)$ by the quadratic $(x-2)(x+3)$. The remainder must have degree less than 2, so write

$$p(x) = (x-2)(x+3)\,q(x) + ax + b.$$

Applying the remainder theorem at $x = 2$ and $x = -3$:

$$p(2) = 2a + b = 7,$$
$$p(-3) = -3a + b = -8.$$

Subtracting the second from the first gives $5a = 15$, so $a = 3$, and then $b = 7 - 2(3) = 1$.

Hence $p(x) = (x-2)(x+3)\,q(x) + 3x + 1$, which is option A.

Check: at $x = 2$, $3(2) + 1 = 7$; at $x = -3$, $3(-3) + 1 = -8$. Both conditions are satisfied.

Question 6

Tags: Graphs of Functions, Polynomial Expansions · Difficulty: 6.5

The curve C is defined by

$$(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)=63.$$

How many points on C have both coordinates integers?

A 0

B 1

C 2

D 3

E 4

F 6

G 8

H infinitely many

Solution 6

Answer: E

Apply the identities $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$ in turn, or otherwise expand:

$$\begin{aligned}(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)&=(x^3+y^3)(x^3-y^3)\\&=x^6-y^6.\end{aligned}$$

So C is the curve $x^6-y^6=63$. starting from $1^6=1$, $2^6=64$, the integer powers of 6 increase in value very rapidly, and difference of consecutive integer powers of 6 increases rapidly, so the only possibilty is $2^6-1^6=63$. Which, when possible signs are taken into account, yields four integer points $(2,1),\ (2,-1),\ (-2,1),\ (-2,-1)$.

So the ansewr is **E**.

Question 7

Tags: Logic Sufficiency · Difficulty: 7

Let $ABCD$ be a convex quadrilateral, with its vertices labelled in order. A quadrilateral is called a rhombus if all four of its sides are equal in length.

Which of the following statements, taken individually, are **sufficient** to guarantee that $ABCD$ is a rhombus?

1. $AB = AD$, and the diagonals AC and BD intersect at right angles.

2. $AB = CD$, and the diagonals AC and BD divide $ABCD$ into four similar triangles.

3. The diagonals AC and BD divide $ABCD$ into four similar triangles.

4. The diagonal AC divides $ABCD$ into two triangles, and the diagonal BD divides $ABCD$ into two triangles, with all four of these triangles having the same area.

5. $AB = BC = CD$, and the triangles ABC and BCD are congruent.

A Only 1.

B Only 2.

C Only 3.

D Only 4.

E Only 5.

F Only 1 and 2.

G Only 2 and 3.

H Only 3 and 4.

I Only 5 and 1.

J None of them.

Solution 7

Answer: F

Statement 1 is not sufficient. For example, take

$$A = (1, 0), \quad B = (0, 1), \quad C = (-2, 0), \quad D = (0, -1).$$

Then the diagonals AC and BD are perpendicular, and $AB = AD = \sqrt{2}$. However, $BC = CD = \sqrt{5}$, so $ABCD$ is not a rhombus.

Now consider statement 2. Let the diagonals meet at O, and write

$$OA = a, \quad OB = b, \quad OC = c, \quad OD = d.$$

Since the four triangles AOB, BOC, COD and DOA are all similar, the triangles AOB and BOC are similar. The angles $\angle AOB$ and $\angle BOC$ are supplementary.

If two similar non-degenerate triangles have supplementary angles at O, those angles must both be 90°. Hence the diagonals are perpendicular.

So the four small triangles are similar right-angled triangles.

Now compare triangles AOB and COD. They are similar right-angled triangles. Also, statement 2 gives $AB = CD$, which are their hypotenuses. Therefore the two triangles are congruent.

Hence either $a = c$ and $b = d$, or $a = d$ and $b = c$.

If $a = c$ and $b = d$, then

$$AB = BC = CD = DA = \sqrt{a^2 + b^2}.$$

So $ABCD$ is a rhombus.

If $a = d$ and $b = c$, then triangle BOC has equal perpendicular sides. Since all four small triangles are similar, all four are isosceles right-angled triangles. Hence again all four sides of $ABCD$ are equal.

So statement 2 is sufficient.

Statement 3 is not sufficient. For example, take

$$A = (1, 0), \quad B = (0, 2), \quad C = (-4, 0), \quad D = (0, -2).$$

The diagonals divide the quadrilateral into four right-angled triangles with side ratio $1:2$, so the four triangles are similar. However, the side lengths of $ABCD$ are

$$\sqrt{5}, \quad \sqrt{20}, \quad \sqrt{20}, \quad \sqrt{5},$$

so it is not a rhombus.

Statement 4 is not sufficient. A non-square rectangle satisfies statement 4, since each diagonal divides the rectangle into two triangles of equal area. However, a non-square rectangle is not a rhombus.

Statement 5 is not sufficient. For example, take

$$A = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad B = (0,0), \quad C = (1,0), \quad D = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right).$$

Then $AB = BC = CD = 1$, and triangles ABC and BCD are congruent, since they are both equilateral triangles. However,

$$AD = \sqrt{3},$$

so $ABCD$ is not a rhombus.

Therefore the only sufficient statement is statement 2.

So the correct answer is **(B) Only 2**.

Question 8

Tags: Logic Sufficiency, Logic Equivalence, Logic Deduction · Difficulty: 7

Let x and y be non-zero real numbers with $x \neq y$.

Which one of the following statements is both **necessary and sufficient** for $x < y$?

A $y^{-1/3} < x^{-1/3}$

B $y^{1/5} < x^{1/5}$

C $x^{2/3} < y^{2/3}$

D $y^{2/3} < x^{2/3}$

E $|x|^{1/3} < |y|^{1/3}$

F $x^{-1/3} < y^{-1/3}$

G $x^{1/5} < y^{1/5}$

Solution 8

Answer: G

The correct answer is **(G)**.

The function $t^{1/5}$ is increasing for all real values of t. Therefore

$$x < y$$

is equivalent to

$$x^{1/5} < y^{1/5}.$$

So (G) is both necessary and sufficient for $x < y$.

We can also eliminate the other options by counterexamples.

For (A), take $x = -1$ and $y = 1$. Then $x < y$, but

$$y^{-1/3} = 1 \quad \text{and} \quad x^{-1/3} = -1,$$

so $y^{-1/3} < x^{-1/3}$ is false. Hence (A) is not necessary.

For (B), since $t^{1/5}$ is increasing, $y^{1/5} < x^{1/5}$ would imply $y < x$, not $x < y$. So (B) is not correct.

For (C), take $x = -2$ and $y = 1$. Then $x < y$, but

$$x^{2/3} = 2^{2/3} > 1 = y^{2/3}.$$

So (C) is not necessary.

For (D), take $x = 1$ and $y = 2$. Then $x < y$, but

$$y^{2/3} > x^{2/3}.$$

So (D) is not necessary.

For (E), take $x = -2$ and $y = 1$. Then $x < y$, but

$$|x|^{1/3} = 2^{1/3} > 1 = |y|^{1/3}.$$

So (E) is not necessary.

For (F), take $x = 1$ and $y = 2$. Then $x < y$, but

$$x^{-1/3} > y^{-1/3}.$$

So (F) is not necessary.

Therefore the only statement which is both necessary and sufficient for $x < y$ is **(G)**.

Question 9

Tags: Logic Deduction, Differentiation · Difficulty: 7

Consider the following statements about the polynomial $\mathrm{p}(x)$, where $a < b$:

(I) $\mathrm{p}(a) < \mathrm{p}(b)$;

(II) $\mathrm{p}'(x) \geq 0$ for all $x \in [a, b]$;

(III) $\int_a^b \mathrm{p}(x)\, dx \geq 0$;

(IV) $\mathrm{p}(a) + \mathrm{p}(b) \geq 2\,\mathrm{p}\left(\frac{a+b}{2}\right)$.

Which of these statements is a **necessary** condition for $\mathrm{p}(x)$ to be increasing for $a \leq x \leq b$? (Here "increasing" means $x_1 \leq x_2 \Rightarrow \mathrm{p}(x_1) \leq \mathrm{p}(x_2)$.)

A none of them

B I only

C II only

D III only

E IV only

F I and II only

G I and III only

H I and IV only

I II and III only

J II and IV only

K III and IV only

L I, II and III only

M I, II and IV only

N I, III and IV only

O II, III and IV only

Solution 9

Answer: C

We test each statement against the definition: p increasing on $[a, b]$ means $x_1 \le x_2 \Rightarrow \mathrm{p}(x_1) \le \mathrm{p}(x_2)$ (so a constant polynomial counts as increasing).

I (NOT necessary). Take $\mathrm{p}(x) = 7$. Then p is (non-strictly) increasing on $[a, b]$, but $\mathrm{p}(a) = 7 = \mathrm{p}(b)$, so $\mathrm{p}(a) < \mathrm{p}(b)$ fails. The strict inequality is not forced by non-strict monotonicity.

II (NECESSARY). p is a polynomial, hence differentiable on $[a, b]$. If $\mathrm{p}'(c) < 0$ for some $c \in [a, b]$, then by the limit definition of the derivative there exist points $x_1 < x_2$ near c with $\mathrm{p}(x_1) > \mathrm{p}(x_2)$, contradicting that p is increasing on $[a, b]$. Hence $\mathrm{p}'(x) \ge 0$ throughout $[a, b]$.

III (NOT necessary). Take $\mathrm{p}(x) = x - 100$ on $[a, b] = [0, 1]$. Then $\mathrm{p}'(x) = 1 > 0$, so p is increasing. But

$$\int_0^1 (x - 100)\, dx = \frac{1}{2} - 100 = -\frac{199}{2} < 0.$$

A vertical shift can make the integral arbitrarily negative without affecting monotonicity.

IV (NOT necessary). The inequality $\mathrm{p}(a) + \mathrm{p}(b) \ge 2\mathrm{p}\left(\frac{a+b}{2}\right)$ is midpoint **convexity**, which is unrelated to monotonicity. Take $\mathrm{p}(x) = -x^2$ on $[a, b] = [-2, -1]$. On this interval $\mathrm{p}'(x) = -2x \in [2, 4] > 0$, so p is strictly increasing. But

$$\mathrm{p}(-2) + \mathrm{p}(-1) = -4 + (-1) = -5, \qquad 2\mathrm{p}\left(-\frac{3}{2}\right) = 2 \cdot \left(-\frac{9}{4}\right) = -\frac{9}{2},$$

and $-5 \ge -\frac{9}{2}$ is false. (Concave increasing functions violate IV.)

Only II is necessary, so the answer is **C**.

Question 10

Tags: Sequences and Series · Difficulty: 7

A lattice path starts at the origin $(0, 0)$ and proceeds in steps of unit length along the integer lattice. The successive leg lengths are $2, 2, 4, 4, 6, 6, 8, 8, 10, 10, \ldots$ (each positive even integer appearing twice in succession), and the directions cycle in the order **Right, Up, Left, Down, Right, Up, Left, Down,** ... Thus the first eight legs of the path are R2, U2, L4, D4, R6, U6, L8, D8, and the path continues indefinitely. Which one of the following lattice points is **not** visited by the path? (Note that a visited lattice point need not be a turning point, it can be any point on the continuous path.)

A $(99, 100)$

B $(-99, 100)$

C $(100, 99)$

D $(-100, 99)$

E $(99, -100)$

F $(100, -99)$

G $(-99, -100)$

H $(-100, -99)$

Solution 10

Answer: F

Trace the path through the first few legs to identify the structure. After the first four legs R2, U2, L4, D4 the position is $(-2, -2)$. The next four legs R6, U6, L8, D8 end at $(-4, -4)$. The pattern continues: after step number $4k$ (a Down-leg) the position is $(-2k, -2k)$. The four legs forming the k-th outer loop are as follows. (i) $\mathrm{R}(4k + 2)$ from $(-2k, -2k)$ to $(2k + 2, -2k)$, covering row $y = -2k$ at $x \in [-2k, 2k + 2]$. (ii) $\mathrm{U}(4k + 2)$ from $(2k + 2, -2k)$ to $(2k + 2, 2k + 2)$, covering column $x = 2k + 2$ at $y \in [-2k, 2k + 2]$. (iii) $\mathrm{L}(4k + 4)$ from $(2k + 2, 2k + 2)$ to $(-2k - 2, 2k + 2)$, covering row $y = 2k + 2$ at $x \in [-2k - 2, 2k + 2]$. (iv) $\mathrm{D}(4k + 4)$ from $(-2k - 2, 2k + 2)$ to $(-2k - 2, -2k - 2)$, covering column $x = -2k - 2$ at $y \in [-2k - 2, 2k + 2]$.

Thus every visited lattice point lies in one of four families (with $k \geq 0$): (i) row $y = -2k$ with $x \in [-2k, 2k + 2]$; (ii) column $x = 2k + 2$ with $y \in [-2k, 2k + 2]$; (iii) row $y = 2k + 2$ with

$x \in [-2k-2, 2k+2]$; (iv) column $x = -2k-2$ with $y \in [-2k-2, 2k+2]$. In particular, every visited point (a, b) has at least one of a, b even.

The candidate $(100, -99)$ has $a = 100$ even and $b = -99$ odd. The only family that could capture it is column $x = 100 = 2k + 2$ with $k = 49$, which covers $y \in [-2k, 2k+2] = [-98, 100]$. Since $-99 < -98$, the point lies just outside this column's range. Row $y = -99$ is not a row visited by the path (rows visited have even y). So $(100, -99)$ is **not** on the path.

Verification of the other seven options: $(99, 100)$ lies in row $y = 100 = 2(49)+2$ with range $[-100, 100]$, which contains 99. $(-99, 100)$ lies in the same row, which contains -99. $(100, 99)$ lies in column $x = 100$ with range $[-98, 100]$, which contains 99. $(-100, 99)$ lies in column $x = -100 = -2(49) - 2$ with range $[-100, 100]$, which contains 99. $(99, -100)$ lies in row $y = -100 = -2(50)$ with range $[-100, 102]$, which contains 99. $(-99, -100)$ lies in the same row, which contains -99. $(-100, -99)$ lies in column $x = -100$ with range $[-100, 100]$, which contains -99. The answer is F.

Question 11

Tags: Logic Sufficiency, Differentiation, Logic Deduction · Difficulty: 7

$f(x)$ is a polynomial function defined for all real x.

Statement P: $f'(5) = 0$ and $f''(5) \geqslant 0$.

Statement Q: f has a local minimum at $x = 5$.

Which one of the following is true?

A P is neither necessary nor sufficient for Q.

B P is necessary but not sufficient for Q.

C P is sufficient but not necessary for Q.

D P is necessary and sufficient for Q.

E P is sufficient for Q, and Q is sufficient for P, but they are not equivalent.

F Whether P is necessary or sufficient for Q depends on the degree of f.

Solution 11

Answer: B

We must test both directions independently.

Does P imply Q? Consider $f(x) = (x-5)^3$. Then $f'(x) = 3(x-5)^2$, so $f'(5) = 0$, and $f''(x) = 6(x-5)$, so $f''(5) = 0 \geqslant 0$. Hence P holds. But $x = 5$ is a stationary point of inflection (since f is increasing on both sides of 5), not a local minimum. So Q fails. Therefore P does **not** imply Q.

Does Q imply P? Suppose f has a local minimum at $x = 5$. Since f is a polynomial, it is differentiable everywhere, and a local extremum of a differentiable function forces $f'(5) = 0$. For the second condition, suppose for contradiction that $f''(5) < 0$. Then by the second derivative test, $x = 5$ would be a local **maximum**, contradicting Q. So $f''(5) \geqslant 0$. Hence P holds.

(Note: we cannot conclude $f''(5) > 0$ from Q. For instance, $f(x) = (x-5)^4$ has a local minimum at $x = 5$ with $f''(5) = 0$.)

So Q implies P but P does not imply Q: P is necessary but not sufficient for Q. Answer **B**.

Question 12

Tags: Logic Deduction · Difficulty: 7.5

In a TMUA-style question, students are asked to determine which, if any, of the five statements $1, 2, 3, 4, 5$ are true. They must then identify the correct option from the list provided in the question.

Assume that a **competent student** can always correctly determine the truth value of any statement they choose to check.

A teacher is designing such a question and wants to choose the option set that **maximises the smallest number of checks** a lucky student might need to determine the correct option. Equivalently, the teacher wants to maximise the fewest checks after which it is *possible* for the student to know the correct option.

Which of the following four candidate option sets best achieves the teacher's aim?

Option Set I

A: only statement 1 is true
B: only statements $1, 2, 3$ are true
C: only statements $1, 2, 4$ are true
D: only statements $1, 4, 5$ are true
E: only statements $1, 5$ are true

Option Set II

A: only statement 1 is true
B: only statement 2 is true
C: only statement 3 is true
D: only statement 4 is true
E: only statement 5 is true

Option Set III

A: only statements $1, 2$ are true
B: only statements $2, 3$ are true
C: only statements $3, 4$ are true
D: only statements $4, 5$ are true
E: no statements are true

Option Set IV

A: only statements $1, 2$ are true
B: only statements $2, 3$ are true
C: only statements $3, 4$ are true
D: only statements $4, 5$ are true
E: only statements $1, 5$ are true

A **Option Set I**

B **Option Set II**

C **Option Set III**

D **Option Set IV**

Solution 12

Answer: D

For each option set, we look for the smallest number of statement checks that could identify an option in the luckiest case.

If one statement check can uniquely identify an option, then the smallest possible number of checks for that option set is 1.

For **Option Set I**, statement 3 is true only in option B. So if a student checks statement 3 and finds it true, they immediately know the answer is B. Therefore the lucky minimum is 1.

For **Option Set II**, each option says exactly one different statement is true. So if the student checks the true statement, they immediately know the answer. Therefore the lucky minimum is 1.

For **Option Set III**, statement 1 is true only in option A, and statement 5 is true only in option D. So one lucky check is enough. Therefore the lucky minimum is 1.

For **Option Set IV**, the options are

$$\begin{array}{ll} \text{A:} & \{1,2\} \\ \text{B:} & \{2,3\} \\ \text{C:} & \{3,4\} \\ \text{D:} & \{4,5\} \\ \text{E:} & \{1,5\} \end{array}$$

Each statement appears in exactly two options. For example, statement 1 is true in A and E, statement 2 is true in A and B, and so on. Therefore no single statement check can ever identify the correct option.

However, two checks can identify an option. For example, if statements 1 and 2 are both true, then the answer must be A. Similarly, each option has a pair of true statements that appears in no other option.

So the lucky minimum for Option Set IV is 2.

Thus the teacher should choose **Option Set IV**, since it maximises the smallest number of checks a lucky student might need.

Question 13

Tags: Logic Deduction, Logic Sufficiency · Difficulty: 7.5

For which of the following is it a **necessary but not sufficient** condition that

$$2x^3 - 3px^2 + p = 0$$

has exactly one real root?

A $-1 < p < 1$

B $-1 \le p \le 1$

C $0 < p < 1$

D $p = 0$

E $p \le -1$

F $p \ge 1$

Solution 13

Answer: B

Let

$$f(x) = 2x^3 - 3px^2 + p.$$

Then

$$f'(x) = 6x^2 - 6px = 6x(x - p).$$

So the stationary points are at $x = 0$ and $x = p$.

Their corresponding values are $f(0) = p$ and

$$f(p) = 2p^3 - 3p^3 + p = p - p^3 = p(1 - p^2).$$

For the cubic to have exactly one real root, the two stationary values must have the same sign. If one of them is zero, then the cubic has a repeated root and another real root.

So we need

$$p \cdot p(1 - p^2) > 0.$$

This gives

$$p^2(1-p^2) > 0.$$

Since $p^2 > 0$ when $p \neq 0$, we need

$$1 - p^2 > 0.$$

So

$$-1 < p < 1.$$

The case $p = 0$ gives $2x^3 = 0$, which has exactly one real root, so $p = 0$ is also included.

Therefore the exact condition is

$$-1 < p < 1.$$

Option (A) is necessary and sufficient, so it is not the answer.

Option (B), $-1 \leq p \leq 1$, is necessary because every value satisfying $-1 < p < 1$ also satisfies $-1 \leq p \leq 1$. However, it is not sufficient because $p = -1$ and $p = 1$ do not give exactly one real root.

Therefore the correct answer is **B**.

Question 14

Tags: Inequalities, General Algebra · Difficulty: 7.5

Let $f(x) = x^3 - px^2 - p^2x + k$ where k is a real constant. Suppose that p and q are real numbers satisfying $-3 \leq p \leq 3$ and $-3 \leq q \leq 3$.

The function f has non-negative gradient at $x = q$.

Find the area of the region of all possible points (p, q) in the p-q plane.

A 20

B 21

C 24

D 30

E 36

F 42

G 56

H 60

Solution 14

Answer: C

We have

$$f'(x) = 3x^2 - 2px - p^2.$$

Since f has non-negative gradient at $x = q$, we need

$$3q^2 - 2pq - p^2 \geq 0.$$

Factorising,

$$3q^2 - 2pq - p^2 = (3q + p)(q - p).$$

So we need

$$(3q + p)(q - p) \geq 0.$$

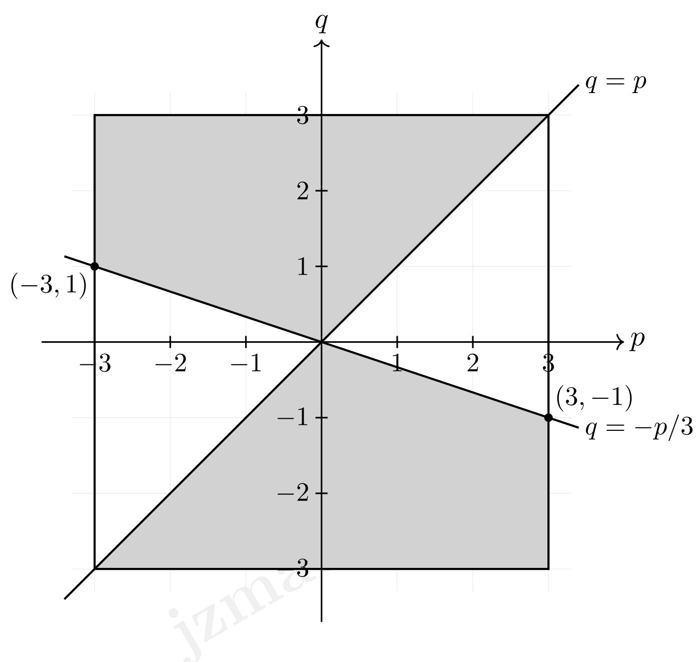
The boundary lines are

$$p = q$$

and

$$p = -3q.$$

We work inside the square $-3 \le p \le 3$, $-3 \le q \le 3$.



The area is 24.

Question 15

Tags: General Algebra · Difficulty: 7.5

Which one of the following numbers is largest in value?

A $\log_2 7$

B $\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$

C $\dfrac{1}{\sqrt{3}-\sqrt{2}}$

D $4^{4/5}$

E $\sqrt{6+2\sqrt{5}}$

Solution 15

Answer: E

Each entry must be put into a comparable form.

A: $\log_2 7 < \log_2 8 = 3$. More precisely $2^{2.8} \approx 6.96$, so $\log_2 7 \approx 2.807$.

B: Geometric series with first term 1 and ratio 2/3:

$$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1}{1-2/3} = 3$$

exactly.

C: Rationalise:

$$\frac{1}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \sqrt{3}+\sqrt{2} \approx 1.732 + 1.414 = 3.146.$$

D: $4^{4/5} = 2^{8/5}$. Compare with 3 by raising both to the 5th power: $2^8 = 256$ vs $3^5 = 243$, so $2^{8/5} > 3$. Numerically $2^{8/5} \approx 3.031$.

E: Recognise $6+2\sqrt{5} = 1+2\sqrt{5}+5 = (1+\sqrt{5})^2$, so $\sqrt{6+2\sqrt{5}} = 1+\sqrt{5} \approx 3.236$.

Ordering: $A(2.807) < B(3) < D(3.031) < C(3.146) < E(3.236)$. Largest is E.

Final pairwise check: C vs E — $C^2 = 5+2\sqrt{6} \approx 9.899$, $E^2 = 6+2\sqrt{5} \approx 10.472$, so $C < E$.

Question 16

Tags: Logic Equivalence, Transformation of Graphs · Difficulty: 7.5

Three transformations of the plane are defined as follows. R is the reflection in the y-axis. T is the translation by 4 units in the positive x-direction. S is the horizontal stretch with scale factor 2 (parallel to the x-axis, fixing the y-axis).

Let $y = \mathrm{f}(x)$ be a function defined on the real numbers.

When the graph of $y = \mathrm{f}(x)$ is transformed by applying R, then T, then S (in that order), the result is the graph of $y = \mathrm{g}(x)$.

When the graph of the same function $y = \mathrm{f}(x)$ is transformed by applying S, then T, then R (in that order), the result is the graph of $y = \mathrm{h}(x)$.

Which one of the following conditions on $y = \mathrm{f}(x)$ is **necessary and sufficient** for the functions $\mathrm{g}(x)$ and $\mathrm{h}(x)$ to be identical?

A $\mathrm{f}(x) = \mathrm{f}(x + 3)$ for all x

B $\mathrm{f}(x) = \mathrm{f}(x + 4)$ for all x

C $\mathrm{f}(x) = \mathrm{f}(x + 6)$ for all x

D $\mathrm{f}(x) = \mathrm{f}(x + 12)$ for all x

E $\mathrm{f}(x) = \mathrm{f}(-x)$ for all x

F $\mathrm{f}(x) = \mathrm{f}(6 - x)$ for all x

G $\mathrm{f}(x) = \mathrm{f}(8 - x)$ for all x

Solution 16

Answer: C

Compute $\mathrm{g}(x)$ by applying R, T, S in turn to $y = \mathrm{f}(x)$.

Apply R (reflect in the y-axis, replace x by $-x$): $y = \mathrm{f}(-x)$.

Apply T (translate by $+4$ in the x-direction, replace x by $x - 4$): $y = \mathrm{f}(-(x - 4)) = \mathrm{f}(4 - x)$.

Apply S (horizontal stretch by factor 2, replace x by $x/2$): $y = \mathrm{f}(4 - x/2)$.

So $g(x) = f(4 - x/2)$.

Now compute $h(x)$ by applying S, T, R in turn to $y = f(x)$.

Apply S: $y = f(x/2)$.

Apply T: $y = f((x - 4)/2) = f(x/2 - 2)$.

Apply R: $y = f(-x/2 - 2)$.

So $h(x) = f(-x/2 - 2)$.

For $g(x) = h(x)$ for all x, we need $f(4 - x/2) = f(-x/2 - 2)$ for all x.

Let $u = -x/2 - 2$. As x ranges over $\mathbb{R}$, so does u, and $4 - x/2 = u + 6$. Hence the condition is $f(u + 6) = f(u)$ for all $u \in \mathbb{R}$, i.e. $f(x) = f(x + 6)$ for all x.

This is necessary (derived from $g = h$) and sufficient (substituting back gives equality). Hence the answer is C.

Question 17

Tags: Logic Deduction, Exponentials and Logarithms · Difficulty: 7.5

Let $a, b, c > 0$. Consider the simultaneous equations:

$$\log_b a = 2, \qquad \log_a c = 3, \qquad \log_b\left(c^2 - 5c + 5\right) = 6.$$

Which one of the following statements about the values of a that satisfy all three equations is correct?

A The equations specify a uniquely.

B The equations are satisfied by exactly two values of a.

C The equations are satisfied by infinitely many values of a.

D The equations are contradictory; no value of a satisfies them.

E The equations are satisfied by exactly three values of a.

Solution 17

Answer: A

We must respect every constraint imposed by the logarithms: each base is in $(0, 1) \cup (1, \infty)$ and each argument is positive. So in addition to $a, b, c > 0$ we require $a \neq 1$, $b \neq 1$, and $c^2 - 5c + 5 > 0$.

From $\log_b a = 2$ we obtain $a = b^2$.

From $\log_a c = 3$ we obtain $c = a^3 = b^6$.

From $\log_b\left(c^2 - 5c + 5\right) = 6$ we obtain $c^2 - 5c + 5 = b^6 = c$.

Hence $c^2 - 6c + 5 = 0$, which factorises as $(c - 1)(c - 5) = 0$, giving $c = 1$ or $c = 5$.

Case $c = 1$. Then $b^6 = 1$ with $b > 0$ forces $b = 1$, which is forbidden as a logarithm base. This case is rejected.

Case $c = 5$. Then $b = 5^{1/6}$, so $a = b^2 = 5^{1/3}$. Check: $b > 0$ and $b \neq 1$; $a > 0$ and $a \neq 1$; and $c^2 - 5c + 5 = 25 - 25 + 5 = 5 > 0$. All constraints hold.

Exactly one valid solution exists, with $a = 5^{1/3}$. The equations specify a uniquely.

Question 18

Tags: Logic Counterexample, Graphs of Functions · Difficulty: 8

A student makes the following **claim**:

If f is a real-valued function such that, for some constant a, $\dfrac{f(x)+f(-x)}{2}=a$ for **all real values** of x, then there exists a constant k such that $f(x)=kx+a$ for **all real values** of x.

Examine their **claim** above, and determine which of the following is true?

A The claim is true.

B $f(x)=2x+1$ is a counterexample to the claim.

C $f(x)=\sin x$ is a counterexample to the claim.

D $f(x)=\tan x$ is a counterexample to the claim.

E $f(x)=\cos x$ is a counterexample to the claim.

F $f(x)=-x$ is a counterexample to the claim.

Solution 18

Answer: C

A counterexample must satisfy the assumption of the claim, but fail the conclusion.

For (B), if $f(x)=2x+1$, then

$$\frac{f(x)+f(-x)}{2}=\frac{(2x+1)+(-2x+1)}{2}=1.$$

So the assumption is true with $a=1$. However, $f(x)=2x+1$ is of the form $kx+a$, with $k=2$ and $a=1$. So (B) is not a counterexample.

For (C), if $f(x)=\sin x$, then

$$\frac{f(x)+f(-x)}{2}=\frac{\sin x+\sin(-x)}{2}=0.$$

So the assumption is true with $a=0$.

However, $\sin x$ is not of the form kx. If $\sin x = kx$ for all real values of x, then using $x = \pi$ gives $0 = k\pi$, so $k = 0$. But using $x = \frac{\pi}{2}$ gives $1 = \frac{k\pi}{2}$, which would require $k = \frac{2}{\pi}$. This is impossible. So (C) is a counterexample.

For (D), $f(x) = \tan x$ is not defined for all real values of x, for example at $x = \frac{\pi}{2}$. Therefore it does not satisfy the assumption of the claim, so it is not a counterexample.

For (E), if $f(x) = \cos x$, then

$$\frac{f(x) + f(-x)}{2} = \frac{\cos x + \cos x}{2} = \cos x.$$

This is not constant, so the assumption is false. Hence (E) is not a counterexample.

For (F), if $f(x) = -x$, then

$$\frac{f(x) + f(-x)}{2} = \frac{-x + x}{2} = 0.$$

So the assumption is true with $a = 0$. However, $f(x) = -x$ is of the form $kx + a$, with $k = -1$ and $a = 0$. So (F) is not a counterexample.

Therefore the correct answer is **(C)**.

Question 19

Tags: General Trigonometry, General Number of Solutions · Difficulty: 8.5

Given that p is an integer, find the minimum value of p such that the equation

$$\sqrt{\frac{1+\sin x}{1-\sin 2x}} = \sqrt{\frac{1+\sin 2x}{1-\sin x}}$$

has exactly 16 solutions in the interval $0^\circ \le x < p^\circ$.

A 540

B 541

C 720

D 721

E 900

F 901

G 1080

H 1081

Solution 19

Answer: F

Solution 1

The equation is equivalent to

$$1 - \sin^2 x = 1 - \sin^2 2x.$$

So

$$\cos^2 x = \cos^2 2x,$$

which is the same as

$$|\cos x| = |\cos 2x|.$$

Now consider the graphs of $y = |\cos x|$ and $y = |\cos 2x|$. Both graphs repeat every 180°, so it is enough to count the intersections in one interval of length 180°.

On $0^\circ \le x < 180^\circ$, the graph of $y = |\cos x|$ makes one V-shape, while $y = |\cos 2x|$ makes two V-shapes. From the graph, they intersect 3 times in each interval of the form

$$180m^\circ \le x < 180(m+1)^\circ.$$

The excluded denominator cases do not remove any of these intersections, because when $\sin x = 1$ or $\sin 2x = 1$, the equation $|\cos x| = |\cos 2x|$ is not satisfied.

Therefore, in every 180° block there are exactly 3 solutions. In the interval

$$0^\circ \le x < 900^\circ,$$

there are 5 complete blocks, so there are

$$5 \cdot 3 = 15$$

solutions.

The next block starts at 900°, and at this point both graphs have value 1, so $x = 900^\circ$ is the next solution. Since the interval is $0^\circ \le x < p^\circ$, we need $p > 900$ to include this solution.

Therefore the minimum integer value of p is **901**.

Solution 2

Let $s = \sin x$ and $t = \sin 2x$. The equation becomes

$$\sqrt{\frac{1+s}{1-t}} = \sqrt{\frac{1+t}{1-s}}.$$

Squaring both sides gives

$$\frac{1+s}{1-t} = \frac{1+t}{1-s}.$$

Cross-multiplying,

$$(1+s)(1-s) = (1+t)(1-t).$$

So

$$1 - s^2 = 1 - t^2.$$

Hence $s^2 = t^2$, which means

$$\sin^2 x = \sin^2 2x.$$

Using $\sin 2x = 2\sin x \cos x$, we get

$$\sin^2 x = 4\sin^2 x \cos^2 x.$$

So

$$\sin^2 x(4\cos^2 x - 1) = 0.$$

Therefore either $\sin x = 0$ or $\cos x = \pm\frac{1}{2}$.

This gives solutions at every multiple of $60°$:

$$0°,\ 60°,\ 120°,\ 180°, \ldots$$

These values do not make $1 - \sin x$ or $1 - \sin 2x$ equal to 0, so they are all valid.

The 16th solution is

$$(16 - 1) \cdot 60° = 900°.$$

Since the interval is $0° \le x < p°$, we need $p > 900$ to include this solution. The next solution is $960°$, so the smallest integer value of p is

901.

Question 20

Tags: Sequences and Series, Logic Deduction · Difficulty: 8.5

An infinite sequence of integers $u_1, u_2, u_3, \dots$ satisfies $u_1 = 10$ and

$$u_{n-1} - 9 < u_n < u_{n-1} - 2 \quad \text{for } n > 1.$$

It is given that $u_k = -50$ and that there exists some integer j with $1 < j < k$ such that $u_j = -10$.

How many different values can k take?

A 4

B 9

C 10

D 12

E 13

F 15

G 36

H 48

Solution 20

Answer: D

Since the u_n are integers and $u_{n-1} - 9 < u_n < u_{n-1} - 2$, the integer values strictly between $u_{n-1} - 9$ and $u_{n-1} - 2$ are $u_{n-1} - 8, u_{n-1} - 7, \dots, u_{n-1} - 3$. So the per-step decrease $d_i = u_{i-1} - u_i$ satisfies $d_i \in \{3, 4, 5, 6, 7, 8\}$.

Split the journey into two phases at the required intermediate value $u_j = -10$.

Phase 1 (from $u_1 = 10$ to $u_j = -10$): total drop $= 20$ over m_1 steps. We need

$$3m_1 \le 20 \le 8m_1,$$

so $m_1 \in \{3,4,5,6\}$. Each value is achievable:

$$m_1 = 3:\ 8+8+4 = 20\ m_1 \quad = 4:\ 5+5+5+5 = 20\ m_1 = 5:\ 4+4+4+4+4 = 20\ m_1 \quad = 6:\ 3+3+3+3$$

Phase 2 (from $u_j = -10$ to $u_k = -50$): total drop $= 40$ over m_2 steps. We need

$$3m_2 \leq 40 \leq 8m_2,$$

so $m_2 \in \{5,6,7,8,9,10,11,12,13\}$. Each value is achievable (mix 3s and 8s, or any intermediate values, to reach the required sum).

The total number of steps is $m = m_1 + m_2$, and $k = m + 1$.

Minimum $m = 3 + 5 = 8$; maximum $m = 6 + 13 = 19$. Because m_1 ranges over four consecutive integers and m_2 ranges over nine consecutive integers, every integer $m \in \{8, 9, \ldots, 19\}$ is achievable.

Hence $k \in \{9, 10, \ldots, 20\}$, giving $\boxed{12}$ possible values.