JZ Mock Set C Paper 1

Solutions

Time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

Average difficulty: 6.825

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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Question 1

Tags: Integration · Difficulty: 5

Find
$$\int_1^4 \frac{(x-2)(x+3)}{x^2\sqrt{x}}\,dx.$$

A $-\dfrac{13}{2}$

B $-\dfrac{5}{2}$

C $-\dfrac{1}{2}$

D $-\dfrac{1}{4}$

E $\dfrac{1}{2}$

F $\dfrac{7}{2}$

G $\dfrac{13}{2}$

Solution 1

Answer: C

First expand the numerator: $(x-2)(x+3) = x^2 + x - 6$. The denominator is $x^2\sqrt{x} = x^{5/2}$, so
$$\frac{(x-2)(x+3)}{x^2\sqrt{x}} = x^{-1/2} + x^{-3/2} - 6x^{-5/2}.$$

Integrating term by term:
$$\int \left(x^{-1/2} + x^{-3/2} - 6x^{-5/2}\right) dx = 2x^{1/2} - 2x^{-1/2} + 4x^{-3/2} + C,$$

since $\int -6x^{-5/2}\,dx = -6 \cdot \frac{x^{-3/2}}{-3/2} = 4x^{-3/2}$.

Evaluate at the limits. At $x = 4$: $2(2) - 2 \cdot \frac{1}{2} + 4 \cdot \frac{1}{8} = 4 - 1 + \frac{1}{2} = \frac{7}{2}$. At $x = 1$: $2 - 2 + 4 = 4$.

Hence the integral equals $\frac{7}{2} - 4 = -\frac{1}{2}$.

Question 2

Tags: Inequalities · Difficulty: 5.5

The real numbers x, y, z satisfy

$$|x+2| \le 5, \qquad |y-1| \le 6, \qquad |z+3| \le 4.$$

What is the greatest possible value of $|xyz|$?

A 21

B 120

C 147

D 245

E 343

F There is no greatest value.

Solution 2

Answer: E

Each absolute-value inequality unpacks to a closed interval:

$$|x+2| \le 5 \iff -7 \le x \le 3,$$

$$|y-1| \le 6 \iff -5 \le y \le 7,$$

$$|z+3| \le 4 \iff -7 \le z \le 1.$$

Since $|xyz| = |x|\,|y|\,|z|$ and the three constraints are independent, $|xyz|$ is maximised by maximising each factor separately.

From the intervals: $|x| \le 7$ (attained at $x = -7$), $|y| \le 7$ (attained at $y = 7$), $|z| \le 7$ (attained at $z = -7$). The point $(x, y, z) = (-7, 7, -7)$ lies in the feasible box, so the bound is achieved.

Hence the greatest value of $|xyz|$ is $7 \times 7 \times 7 = 343$.

Verification by corners. Because $|xyz|$ is a product of absolute values of independent linear quantities, its maximum on a box is attained at a vertex. The eight vertices give $|xyz|$ values

$245,\ 35,\ 343,\ 49,\ 105,\ 15,\ 147,\ 21$, confirming 343 as the maximum.

Question 3

Tags: Inequalities, General Algebra · Difficulty: 5.5

Which of the following five values is the **largest**?

A $\dfrac{7}{5}$

B $\dfrac{17}{12}$

C $\dfrac{10}{7}$

D $\dfrac{11}{8}$

E $2\cos\left(\dfrac{\pi}{4}\right)$

Solution 3

Answer: C

First note that

$$2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}.$$

We compare the other values with $\frac{10}{7}$. Since

$$\frac{10}{7} > \frac{7}{5}$$

because $10 \cdot 5 > 7 \cdot 7$, option C is larger than option A. Also

$$\frac{10}{7} > \frac{17}{12}$$

because $10 \cdot 12 > 17 \cdot 7$, so option C is larger than option B. Similarly,

$$\frac{10}{7} > \frac{11}{8}$$

because $10 \cdot 8 > 11 \cdot 7$, so option C is larger than option D. Finally, since both numbers are positive, we can compare squares:

$$\left(\frac{10}{7}\right)^2 = \frac{100}{49} > 2.$$

Therefore $\frac{10}{7} > \sqrt{2}$, so option C is also larger than option E. Hence the largest value is **C**.

Question 4

Tags: Exponentials and Logarithms · Difficulty: 6

Given the simultaneous equations

$$\log_{10} 2 + \log_{10}(y-1) = \log_{10} x + \log_{10}(x-2),$$

$$\log_{10}(y+3-3x) = 0,$$

the value(s) of y are

A $10 \pm 3\sqrt{10}$

B $10 + 3\sqrt{10}$ only

C $10 - 3\sqrt{10}$ only

D $4 + \sqrt{10}$ only

E $10 \pm \sqrt{10}$

Solution 4

Answer: B

From the first equation, using $\log a + \log b = \log(ab)$ on each side, $2(y-1) = x(x-2)$, valid only when $y > 1$, $x > 0$ and $x > 2$ (so that all four logarithms are defined).

From the second equation, $y + 3 - 3x = 10^0 = 1$, hence $y = 3x - 2$, i.e. $x = (y+2)/3$. Substituting into $2(y-1) = x^2 - 2x$ and multiplying through by 9:

$$18y - 18 = (y+2)^2 - 6(y+2) = y^2 - 2y - 8,$$

so $y^2 - 20y + 10 = 0$ and $y = 10 \pm 3\sqrt{10}$.

Now $\sqrt{10} \approx 3.16$, so the two algebraic candidates are $y \approx 19.49$ and $y \approx 0.51$. The smaller candidate fails the domain condition $y > 1$ (and the corresponding $x = 4 - \sqrt{10} \approx 0.84$ also fails $x > 2$), so it must be rejected. Only $y = 10 + 3\sqrt{10}$ is admissible.

Question 5

Tags: Sequences and Series, Ratio and Proportion · Difficulty: 6

Three players A, B and C take turns throwing a fair six-sided die in the order A, B, C, A, B, C, ... until one of them wins. The win conditions on a player's own throw are:

$$\begin{aligned}&A \text{ wins if } A \text{ throws a 6,}\\&B \text{ wins if } B \text{ throws a 5 or a 6,}\\&C \text{ wins if } C \text{ throws a } 4, 5 \text{ or 6.}\end{aligned}$$

If a player throws and does not win, play passes to the next player. What is the probability that B is the first to win?

A $\frac{3}{13}$

B $\frac{5}{18}$

C $\frac{5}{13}$

D $\frac{6}{13}$

E $\frac{1}{3}$

F $\frac{5}{9}$

Solution 5

Answer: C

Consider one full round (the three throws by A, then B, then C).

The probability that B wins on the first round is the probability that A fails, then B succeeds:

$$P(B \text{ wins in round 1}) = \frac{5}{6} \cdot \frac{2}{6} = \frac{5}{18}.$$

The probability that no one wins in a round is

$$P(\text{no winner in a round}) = \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} = \frac{60}{216} = \frac{5}{18}.$$

If no one wins in a round, the situation resets identically. So

$$P(B \text{ wins}) = \frac{5}{18} + \frac{5}{18} \cdot \frac{5}{18} + \frac{5}{18} \cdot \left(\frac{5}{18}\right)^2 + \cdots$$

This is a geometric series with first term $\frac{5}{18}$ and common ratio $\frac{5}{18}$:

$$P(B \text{ wins}) = \frac{\frac{5}{18}}{1 - \frac{5}{18}} = \frac{\frac{5}{18}}{\frac{13}{18}} = \frac{5}{13}.$$

Question 6

Tags: Geometry, General Algebra · Difficulty: 6

The curve C has equation

$$y = x^2 + bx + 3$$

where $b \geq 0$. Let S denote the stationary point of C, and let O denote the origin.

Find the minimum value of

$$|OS|^2 + b^2$$

as b varies over $b \geq 0$.

A $\frac{11}{4}$

B $\frac{35}{4}$

C $\frac{39}{4}$

D $\frac{51}{4}$

E 9

Solution 6

Answer: B

The stationary point of $y = x^2 + bx + 3$ satisfies $\frac{dy}{dx} = 2x + b = 0$, so $x = -\frac{b}{2}$. Substituting this into the equation of the curve gives

$$y = \left(-\frac{b}{2}\right)^2 + b\left(-\frac{b}{2}\right) + 3 = 3 - \frac{b^2}{4}.$$

Hence

$$S = \left(-\frac{b}{2}, 3 - \frac{b^2}{4}\right).$$

Therefore

$$|OS|^2 = \left(-\frac{b}{2}\right)^2 + \left(3 - \frac{b^2}{4}\right)^2 = \frac{b^2}{4} + \left(3 - \frac{b^2}{4}\right)^2.$$

Let $u = \frac{b^2}{4}$. Since $b \geq 0$, we have $u \geq 0$, and $b^2 = 4u$. Then

$$|OS|^2 + b^2 = u + (3 - u)^2 + 4u.$$

So

$$|OS|^2 + b^2 = u^2 - u + 9.$$

Completing the square,

$$u^2 - u + 9 = \left(u - \frac{1}{2}\right)^2 + \frac{35}{4}.$$

Since $\left(u - \frac{1}{2}\right)^2 \geq 0$, the minimum value is $\frac{35}{4}$. This is possible because $u = \frac{1}{2}$ is in the allowed range $u \geq 0$.

Therefore the minimum value is $\frac{35}{4}$, so the answer is **B**.

Question 7

Tags: Integration, Sequences and Series · Difficulty: 6.5

The two functions $F(n)$ and $G(n)$ are defined as follows for positive integers n:

$$F(n) = \int_0^{2n} \frac{|n-x|}{n}\,dx$$

$$G(n) = \sum_{r=1}^{n} F(r)$$

What is the largest positive integer n such that $G(n) < 210$?

A 15

B 16

C 19

D 20

E 21

Solution 7

Answer: C

We have

$$F(n) = \int_0^{2n} \frac{|n-x|}{n}\,dx.$$

Split the integral at $x = n$:

$$F(n) = \int_0^{n} \frac{n-x}{n}\,dx + \int_n^{2n} \frac{x-n}{n}\,dx.$$

The first part is the area of a triangle with base n and height 1, so it is $\frac{n}{2}$. The second part is also the area of a triangle with base n and height 1, so it is also $\frac{n}{2}$. Therefore

$$F(n) = \frac{n}{2} + \frac{n}{2} = n.$$

So

$$G(n) = \sum_{r=1}^{n} F(r) = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

We need

$$\frac{n(n+1)}{2} < 210.$$

So

$$n(n+1) < 420.$$

Check nearby values:

$$19 \cdot 20 = 380 < 420,$$

but

$$20 \cdot 21 = 420,$$

which is not less than 420. Therefore the largest positive integer is **19**, and so the answer is **C**.

Question 8

Tags: Integration, General Algebra · Difficulty: 6.5

Given that

$$\int_0^1 \left(a + 3b\sqrt{x} + 2cx\right)\, dx = 5 \quad \text{and} \quad \int_0^1 \left(4a + 9b\sqrt{x} + 4cx\right)\, dx = 16,$$

find the value of $a + b$.

A 0

B 1

C 2

D 3

E 4

F 5

Solution 8

Answer: D

Evaluate the first integral. Since $\int_0^1 \sqrt{x}\, dx = \frac{2}{3}$ and $\int_0^1 x\, dx = \frac{1}{2}$, we get

$$\int_0^1 \left(a + 3b\sqrt{x} + 2cx\right)\, dx = a + 3b \cdot \frac{2}{3} + 2c \cdot \frac{1}{2} = a + 2b + c.$$

So

$$a + 2b + c = 5.$$

Evaluate the second integral:

$$\int_0^1 \left(4a + 9b\sqrt{x} + 4cx\right)\, dx = 4a + 9b \cdot \frac{2}{3} + 4c \cdot \frac{1}{2} = 4a + 6b + 2c.$$

So

$$4a + 6b + 2c = 16.$$

Now double the first equation to get $2a + 4b + 2c = 10$. Subtract this from the second equation:

$$(4a + 6b + 2c) - (2a + 4b + 2c) = 16 - 10.$$

Therefore $2a + 2b = 6$, so $a + b = 3$. The answer is **D**.

Question 9

Tags: Transformation of Graphs · Difficulty: 6.5

The curve $x = (y-2)^2 - 1$ is transformed as follows:

First, it is translated vertically in the positive y-direction by 2 units.

Next, it is reflected in the line $x = 2$.

Finally, it is rotated 90° clockwise about the point $(1, 1)$.

Which of the following is the equation of the resulting curve?

A $y = x^2 - 8x + 13$

B $y = x^2 - 8x + 19$

C $y = -x^2 + 8x - 13$

D $x = y^2 - 8y + 15$

E $y = x^2 - 4x + 1$

Solution 9

Answer: A

The original curve is

$$x = (y-2)^2 - 1.$$

This is a sideways quadratic. Its apex is $(-1, 2)$, and it faces right, since x is equal to a positive square expression.

After translating vertically in the positive y-direction by 2 units, the apex moves from $(-1, 2)$ to $(-1, 4)$. The curve still faces right.

Next, the curve is reflected in the line $x = 2$. The apex $(-1, 4)$ is reflected to $(5, 4)$, because -1 is 3 units to the left of $x = 2$, so its reflection is 3 units to the right of $x = 2$. The curve now faces left.

Finally, the curve is rotated 90° clockwise about $(1, 1)$. Relative to $(1, 1)$, the apex $(5, 4)$ has displacement

$$(4, 3).$$

A 90° clockwise rotation sends displacement (u, v) to $(v, -u)$, so

$$(4, 3) \mapsto (3, -4).$$

Therefore the new apex is

$$(1, 1) + (3, -4) = (4, -3).$$

The curve was facing left before the rotation. A direction pointing left becomes a direction pointing up after a 90° clockwise rotation. A quadratic facing up has the form

$$y = (x - h)^2 + k,$$

where (h, k) is its apex.

Since the new apex is $(4, -3)$, the resulting curve is

$$y = (x - 4)^2 - 3.$$

Expanding,

$$y = x^2 - 8x + 16 - 3 = x^2 - 8x + 13.$$

Therefore the correct answer is

$$\textbf{(A)}\ y = x^2 - 8x + 13.$$

Question 10

Tags: Trig Equation Number of Solutions, General Trigonometry · Difficulty: 7

What is the smallest positive value of a for which the line $x = a$ is a line of symmetry of **both** of the curves

$$y = \sin\left(2x - \frac{\pi}{3}\right) \quad \text{and} \quad y = \cos\left(3x + \frac{\pi}{4}\right)?$$

A $\frac{\pi}{12}$

B $\frac{\pi}{4}$

C $\frac{5\pi}{12}$

D $\frac{7\pi}{12}$

E $\frac{11\pi}{12}$

F $\frac{23\pi}{12}$

Solution 10

Answer: E

A vertical line $x = a$ is a line of symmetry of $y = \sin(\theta(x))$ or $y = \cos(\theta(x))$ exactly when the function attains ± 1 at $x = a$ (its peaks and troughs).

First curve. $\sin(2x - \pi/3) = \pm 1$ requires $2x - \pi/3 = \pi/2 + k\pi$ for integer k, giving

$$x = \frac{5\pi}{12} + \frac{k\pi}{2}.$$

Second curve. $\cos(3x + \pi/4) = \pm 1$ requires $3x + \pi/4 = m\pi$ for integer m, giving

$$x = -\frac{\pi}{12} + \frac{m\pi}{3}.$$

Common solutions. Equate:

$$\frac{5\pi}{12} + \frac{k\pi}{2} = -\frac{\pi}{12} + \frac{m\pi}{3}.$$

Multiplying through by $12/\pi$: $5 + 6k = -1 + 4m$, i.e. $2m - 3k = 3$.

Integers (k, m): a particular solution is $(k, m) = (1, 3)$, and the general solution is $(k, m) = (1 + 2t, 3 + 3t)$ for integer t. Substituting back,

$$x = \frac{5\pi}{12} + \frac{(1 + 2t)\pi}{2} = \frac{11\pi}{12} + t\pi.$$

The positive values are $\frac{11\pi}{12}, \frac{23\pi}{12}, \ldots$, so the smallest is $\frac{11\pi}{12}$.

Check. At $x = 11\pi/12$: $2x - \pi/3 = 11\pi/6 - 2\pi/6 = 3\pi/2$, so $\sin = -1$. And $3x + \pi/4 = 11\pi/4 + \pi/4 = 3\pi$, so $\cos = -1$. Both extrema, as required.

Question 11

Tags: Circle Geometry, Geometry · Difficulty: 7

The circle C_1 has equation $x^2 + y^2 = 4$. The circle C_2 has equation $(x-9)^2 + y^2 = 25$. The straight line l is a common tangent to C_1 and C_2 such that both circles lie on the same side of l, and l has positive gradient. Find this gradient.

A $\frac{1}{3}$

B $2\sqrt{2}$

C $\frac{\sqrt{2}}{4}$

D $\frac{3\sqrt{2}}{8}$

E $\frac{\sqrt{2}}{2}$

F $\frac{2\sqrt{2}}{3}$

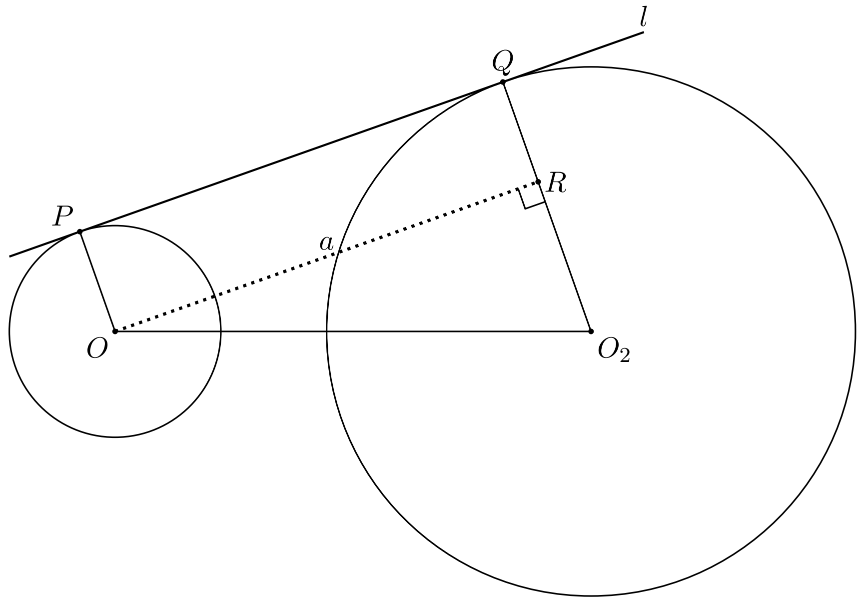
G $\frac{3\sqrt{2}}{4}$

H $\frac{7\sqrt{2}}{8}$

Solution 11

Answer: C

Let $O = (0,0)$ be the centre of C_1, and let $O_2 = (9,0)$ be the centre of C_2.



Since l is tangent to both circles, OP and O_2Q are perpendicular to l. Their lengths are the radii of the circles, so $OP = 2$ and $O_2Q = 5$.

Draw OR parallel to l, and draw O_2R perpendicular to OR, as in the diagram. Then O_2R represents the difference between the perpendicular distances from the two centres to the tangent line, so

$$O_2R = 5 - 2 = 3.$$

Let $OR = a$. Since $OR \perp O_2R$, triangle OO_2R is right-angled at R. Also $OO_2 = 9$, because the centres are $(0, 0)$ and $(9, 0)$.

By Pythagoras,

$$a^2 + 3^2 = 9^2.$$

So

$$a^2 = 72,$$

hence

$$a = 6\sqrt{2}.$$

Now let θ be the angle that l makes with the positive x-axis. Since OR is parallel to l, the gradient of l is $\tan\theta$. In triangle OO_2R,

$$\tan\theta = \frac{O_2R}{OR} = \frac{3}{6\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Therefore the gradient of l is $\frac{\sqrt{2}}{4}$.

Question 12

Tags: Exponentials and Logarithms, General Algebra · Difficulty: 7

Find the minimum value of the function $f(x) = 4^x + 4^{-x} - 3\left(2^x + 2^{-x}\right) + 5$.

A -3

B -1

C $\frac{3}{4}$

D 1

E 3

F 5

Solution 12

Answer: D

First note that

$$\frac{(2^x - 1)^2}{2^x} \geq 0 \quad \Rightarrow \quad 2^x - \frac{1}{2^x} \geq 0.$$

Let $u = 2^x + 2^{-x}$ so then $u \geq 0$. Note that

$$4^x + 4^{-x} = \left(2^x + 2^{-x}\right)^2 - 2 = u^2 - 2.$$

Hence

$$f = u^2 - 2 - 3u + 5 = u^2 - 3u + 3 = \left(u - \frac{3}{2}\right)^2 + \frac{3}{4}.$$

The unconstrained minimiser $u = \frac{3}{2}$ **lies outside** the achievable range $u \geq 2$. Since the parabola opens upward with vertex at $u = \frac{3}{2} < 2$, the quadratic is strictly increasing on $[2, \infty)$. The minimum on $u \geq 2$ is therefore at the boundary $u = 2$:

$$f_{\min} = 2^2 - 3(2) + 3 = 1,$$

achieved at $x = 0$. The minimum value is **1**.

Question 13

Tags: Differentiation, General Number of Solutions · Difficulty: 7

The function f is defined for all real x by

$$f(x) = \frac{1}{4}x^4 - \frac{1}{3}mx^3 - m^2x^2 + n,$$

where m and n are real constants with $m > 0$.

Given that the equation $f(x) = 0$ has four distinct real roots, what is the range of values of n in terms of m?

A $0 < n < \frac{5m^4}{12}$

B $0 < n < \frac{8m^4}{3}$

C $-\frac{5m^4}{12} < n < 0$

D $-\frac{8m^4}{3} < n < 0$

E $-\frac{8m^4}{3} < n < \frac{5m^4}{12}$

F $\frac{5m^4}{12} < n < \frac{8m^4}{3}$

G $0 < n < \frac{m^4}{4}$

H No such n exists

Solution 13

Answer: A

Differentiate:

$$f'(x) = x^3 - mx^2 - 2m^2x = x(x - 2m)(x + m).$$

So the stationary points are at $x = -m$, $x = 0$ and $x = 2m$. Since $m > 0$, they occur in this order. The sign of $f'(x)$ on the four intervals is $-, +, -, +$, so $x = -m$ is a local minimum, $x = 0$ is a local maximum, and $x = 2m$ is a local minimum.

Also, since the coefficient of x^4 is positive, $f(x) \to \infty$ as $x \to \pm\infty$. Therefore, for $f(x) = 0$ to have four distinct real roots, both local minima must be below the x-axis and the local maximum must be above the x-axis.

Now $f(0) = n$, so we need $n > 0$. Also,

$$f(-m) = \frac{1}{4}m^4 + \frac{1}{3}m^4 - m^4 + n = n - \frac{5}{12}m^4$$

so we need $n < \frac{5}{12}m^4$. Finally,

$$f(2m) = 4m^4 - \frac{8}{3}m^4 - 4m^4 + n = n - \frac{8}{3}m^4$$

so we need $n < \frac{8}{3}m^4$. This condition is weaker than $n < \frac{5}{12}m^4$, so it does not change the range.

Therefore the required range is

$$0 < n < \frac{5}{12}m^4,$$

and the answer is **A**.

Question 14

Tags: Sequences and Series, Polynomial Expansions · Difficulty: 7

A positive integer k is chosen uniformly at random from $1, 2, 3, \ldots, 8$.

For each non-negative integer n, define

$$a_n = (2k+1)\sum_{r=0}^{n}\binom{n}{r}\left(-\frac{4-k}{4}\right)^r.$$

Let

$$S = a_0 + a_1 + a_2 + \cdots.$$

What is the probability that S converges to a finite value and $S > 8$?

A $\frac{1}{8}$

B $\frac{1}{4}$

C $\frac{3}{8}$

D $\frac{1}{2}$

E $\frac{5}{8}$

F $\frac{3}{4}$

G $\frac{7}{8}$

H 1

I 0

Solution 14

Answer: B

By the binomial theorem,

$$\sum_{r=0}^{n} \binom{n}{r} \left(-\frac{4-k}{4}\right)^r = \left(1 - \frac{4-k}{4}\right)^n = \left(\frac{k}{4}\right)^n.$$

So $a_n = (2k+1)\left(\frac{k}{4}\right)^n$, and hence

$$S = (2k+1)\left(1 + \frac{k}{4} + \left(\frac{k}{4}\right)^2 + \cdots\right).$$

This converges when $\frac{k}{4} < 1$, so $k < 4$. Since k is a positive integer, the possible values are $k = 1, 2, 3$.

For these values,

$$S = \frac{2k+1}{1-\frac{k}{4}} = \frac{4(2k+1)}{4-k}.$$

We need $S > 8$, so

$$\frac{4(2k+1)}{4-k} > 8.$$

Since $k < 4$, we can multiply by $4-k$ without changing the inequality. This gives $4(2k+1) > 8(4-k)$, so $8k + 4 > 32 - 8k$, hence $16k > 28$. Therefore $k > \frac{7}{4}$.

Among $k = 1, 2, 3$, this gives $k = 2, 3$. Therefore there are 2 successful values out of 8, so the required probability is $\frac{1}{4}$. The answer is **B**.

Question 15

Tags: General Trigonometry, Graphs of Functions · Difficulty: 7.5

The set of points satisfying

$$\sin(x^2 + y^2) = \frac{1}{2}, \qquad x^2 + y^2 \leq 12\pi,$$

is made up of finitely many separate closed curves. For each separate closed curve, consider the region enclosed by that curve. Find the sum of the areas of these regions.

A $31\pi^2$

B $45\pi^2$

C $60\pi^2$

D $66\pi^2$

E $78\pi^2$

F $144\pi^2$

Solution 15

Answer: D

Step 1 (reduce to a 1D level-set). Set $u = x^2 + y^2 \geq 0$. The equation becomes $\sin u = \frac{1}{2}$ with $0 \leq u \leq 12\pi$. Each solution $u = c$ contributes the circle $x^2 + y^2 = c$, i.e. a circle of radius $\sqrt{c}$ centred at the origin, enclosing a disk of area πc.

Step 2 (enumerate u). The general solution of $\sin u = \frac{1}{2}$ is

$$u = \frac{\pi}{6} + 2k\pi \qquad \text{or} \qquad u = \frac{5\pi}{6} + 2k\pi, \quad k \in \mathbb{Z}.$$

We need $0 \leq u \leq 12\pi$, so $k = 0, 1, 2, 3, 4, 5$ for both branches, giving 12 values:

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \frac{37\pi}{6}, \frac{41\pi}{6}, \frac{49\pi}{6}, \frac{53\pi}{6}, \frac{61\pi}{6}, \frac{65\pi}{6}.$$

(The next pair would be $\frac{73\pi}{6} > 12\pi$, excluded; and any $u < 0$ is excluded since $u \geq 0$.)

Step 3 (sum). Pair each $\frac{\pi}{6} + 2k\pi$ with $\frac{5\pi}{6} + 2k\pi$:

$$\left(\frac{\pi}{6} + 2k\pi\right) + \left(\frac{5\pi}{6} + 2k\pi\right) = \pi + 4k\pi.$$

Summing for $k = 0, \ldots, 5$:

$$\sum_{k=0}^{5} (\pi + 4k\pi) = 6\pi + 4\pi(0 + 1 + 2 + 3 + 4 + 5) = 6\pi + 60\pi = 66\pi.$$

Hence the total area is

$$\sum \pi u = \pi \cdot 66\pi = 66\pi^2.$$

Question 16

Tags: Sequences and Series, General Algebra · Difficulty: 7.5

A sequence is defined by

$$u_n = 2\sum_{r=0}^{n-1} u_r \quad \text{for } n \geq 1, \qquad u_0 = 3.$$

Evaluate

$$\sum_{r=0}^{\infty}\left(\frac{1}{u_r} + \frac{1}{u_{r+1}}\right).$$

A the sum does not converge

B $\frac{7}{12}$

C $\frac{5}{6}$

D $\frac{7}{6}$

E $\frac{1}{2}$

Solution 16

Answer: C

Step 1: find a closed form for u_n.

For $n \geq 2$, write $u_n = 2\sum_{r=0}^{n-1} u_r$ and $u_{n-1} = 2\sum_{r=0}^{n-2} u_r$. Subtracting gives $u_n - u_{n-1} = 2u_{n-1}$, so $u_n = 3u_{n-1}$ for $n \geq 2$.

From the definition, $u_1 = 2u_0 = 6$. Hence $u_n = 6 \cdot 3^{n-1}$ for $n \geq 1$, while $u_0 = 3$.

Step 2: split the target sum.

$$\sum_{r=0}^{\infty}\left(\frac{1}{u_r} + \frac{1}{u_{r+1}}\right) = \sum_{r=0}^{\infty}\frac{1}{u_r} + \sum_{r=0}^{\infty}\frac{1}{u_{r+1}}.$$

The second sum re-indexes ($s = r + 1$) to $\sum_{s=1}^{\infty} \frac{1}{u_s}$.

Step 3: evaluate $\sum_{r=1}^{\infty} \frac{1}{u_r}$.

This is geometric with first term $\frac{1}{6}$ and common ratio $\frac{1}{3}$:

$$\sum_{r=1}^{\infty} \frac{1}{u_r} = \frac{1/6}{1-1/3} = \frac{1/6}{2/3} = \frac{1}{4}.$$

Step 4: combine.

$$\sum_{r=0}^{\infty} \frac{1}{u_r} = \frac{1}{u_0} + \frac{1}{4} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

$$\sum_{r=0}^{\infty} \frac{1}{u_{r+1}} = \sum_{s=1}^{\infty} \frac{1}{u_s} = \frac{1}{4}.$$

Total: $\frac{7}{12} + \frac{1}{4} = \frac{7}{12} + \frac{3}{12} = \frac{10}{12} = \frac{5}{6}$. So the answer is **C**.

Question 17

Tags: General Trigonometry, Geometry · Difficulty: 7.5

A triangle ABC is to be drawn with the following measurements:

$$AB = 3a, \qquad BC = \sqrt{5}\,a, \qquad \angle BAC = \theta^\circ$$

where $a > 0$ and $0 < \theta < 90$.

There are two possible triangles consistent with these measurements. Given the area of the larger triangle is 9 times that of the smaller triangle, what is the value of $\cos^2\theta$?

A $\frac{4}{9}$

B $\frac{4}{7}$

C $\frac{16}{27}$

D $\frac{2}{3}$

E $\frac{16}{21}$

F $\frac{64}{81}$

Solution 17

Answer: C

Area ratio iof $9:1$ implies bases ratio must be $3:1$, if we take the different ACs as bases, therefore we can set $AC = x$ or $AC = 3x$.

Next observe that value of $\cos\theta$ is the same if we rescale the triangles, thus we can use $AB = 3$ and $BC = \sqrt{5}$.

Now apply cosine rule to both triangles:

$$5 = 3^2 + x^2 - 6x\cos\theta \quad \text{and} \quad 5 = 3^2 + 9x^2 - 18x\cos\theta.$$

Subtract them and get:

$$x = \frac{3}{2}\cos\theta.$$

Substitute back into any one of the cosine rule equations to eliminate x, you will find $\cos^2\theta = \frac{16}{25}$.
So the answer is **C**.

Question 18

Tags: Differentiation, General Functions, General Algebra · Difficulty: 8

A rectangle has its sides parallel to the coordinate axes and lies entirely within the region enclosed by $y = 2|x-4|$ and $y = x+6$.

What is the maximum possible area of the rectangle?

A $\dfrac{40}{3}$

B 16

C 16.5

D $\dfrac{49}{3}$

E 15

F $\dfrac{50}{3}$

G $\dfrac{35}{2}$

Solution 18

Answer: F

Let the bottom side of the rectangle have equation $y = a$. On the lower curve,

$$a = 2|x-4|.$$

So

$$|x-4| = \frac{a}{2},$$

giving

$$x = 4 - \frac{a}{2} = \frac{8-a}{2}$$

or

$$x = 4 + \frac{a}{2} = \frac{8+a}{2}.$$

Therefore the maximum possible width at height $y = a$ is

$$\frac{8+a}{2} - \frac{8-a}{2} = a.$$

Now take the left bottom corner to be

$$\left(\frac{8-a}{2}, a\right).$$

The top of the rectangle is limited by the line $y = x + 6$. Substituting $x = \frac{8-a}{2}$ gives

$$y = \frac{8-a}{2} + 6 = 10 - \frac{a}{2}.$$

So the height of the rectangle is

$$10 - \frac{a}{2} - a = 10 - \frac{3a}{2}.$$

Hence the area is

$$A = a\left(10 - \frac{3a}{2}\right) = 10a - \frac{3a^2}{2}.$$

Differentiate:

$$A' = 10 - 3a.$$

For a maximum, set $A' = 0$, giving $a = \frac{10}{3}$. Since $A'' = -3 < 0$, this gives a maximum.

Therefore the maximum area is

$$A = \frac{10}{3}\left(10 - \frac{3}{2} \cdot \frac{10}{3}\right) = \frac{10}{3} \cdot 5 = \frac{50}{3}.$$

So the maximum possible area is $\frac{50}{3}$. The answer is **F**.

Question 19

Tags: General Trigonometry, Graphs of Functions · Difficulty: 8

The function f is defined for all real x by $f(x) = \cos\big(\pi\ \sin(\cos x)\big)$.

Let P be the smallest positive period of f, and let M be the minimum value attained by f. Which one of the following is correct?

A $P = \frac{\pi}{2}$ and $M = -1$

B $P = \frac{\pi}{2}$ and $M = \cos(\pi \sin 1)$

C $P = \pi$ and $M = -1$

D $P = \pi$ and $M = \cos(\pi \sin 1)$

E $P = \pi$ and $M = -\cos(\pi \sin 1)$

F $P = 2\pi$ and $M = -1$

G $P = 2\pi$ and $M = \cos(\pi \sin 1)$

H $P = 2\pi$ and $M = \cos 1$

Solution 19

Answer: D

Period. Using $\cos(x + \pi) = -\cos x$ and the fact that sin is odd, then that cos is even:

$$\begin{aligned} f(x+\pi) &= \cos\big(\pi \sin(\cos(x+\pi))\big) \\ &= \cos\big(\pi \sin(-\cos x)\big) \\ &= \cos\big(-\pi \sin(\cos x)\big) \\ &= \cos\big(\pi \sin(\cos x)\big) = f(x). \end{aligned}$$

So π is a period. To check it is the smallest: $f(0) = \cos(\pi \sin 1)$ and $f(\pi/2) = \cos(\pi \sin 0) = \cos 0 = 1$. Since $\cos(\pi \sin 1) \neq 1$, we have $f(0) \neq f(\pi/2)$, so $\pi/2$ is not a period. Hence $P = \pi$.

Minimum. As x varies over $\mathbb{R}$, $\cos x$ ranges over $[-1, 1]$. Then $\sin(\cos x)$ ranges over $[-\sin 1, \sin 1]$ since sin is increasing on $[-1, 1] \subset [-\pi/2, \pi/2]$. Thus $u = \pi \sin(\cos x)$ ranges over $[-\pi \sin 1, \pi \sin 1]$.

Now $\sin 1 < 1$, so $\pi \sin 1 < \pi$; also $\sin 1 > \sin(\pi/6) = 1/2$, so $\pi \sin 1 > \pi/2$. Hence u ranges over an interval contained in $(-\pi, \pi)$ and reaching past $\pm\pi/2$.

On $[-\pi \sin 1, \pi \sin 1]$, $\cos u$ attains its minimum at the endpoints (where $|u|$ is largest), giving

$$M = \cos(\pi \sin 1).$$

Since $\pi/2 < \pi \sin 1 < \pi$, this value lies in $(-1, 0)$ — strictly greater than -1. The endpoint $u = \pm\pi \sin 1$ is achieved when $\cos x = \pm 1$, i.e. at $x = 0$ and $x = \pi$.

Question 20

Tags: Exponentials and Logarithms, Polynomial Expansions · Difficulty: 8.5

Given that

$$f(x) = \ln\left(\frac{1+x}{1-x}\right), \quad \text{where } -1 < x < 1,$$

find constants a and b such that

$$f\left(\frac{u+v}{1+uv}\right) = af(u) + bf(v).$$

Hence, or otherwise, determine which of the following is equal to

$$f\left(\frac{3x+x^3}{1+3x^2}\right) - f\left(\frac{2x}{1+x^2}\right).$$

A $-2f(x)$

B $-f(x)$

C $f(x)$

D $2f(x)$

E $3f(x)$

F 0

Solution 20

Answer: C

First find a and b. We have

$$f\left(\frac{u+v}{1+uv}\right) = \ln\left(\frac{1+\frac{u+v}{1+uv}}{1-\frac{u+v}{1+uv}}\right).$$

Simplifying the fraction gives

$$f\left(\frac{u+v}{1+uv}\right) = \ln\left(\frac{(1+u)(1+v)}{(1-u)(1-v)}\right).$$

So

$$f\left(\frac{u+v}{1+uv}\right) = \ln\left(\frac{1+u}{1-u}\right) + \ln\left(\frac{1+v}{1-v}\right) = f(u) + f(v).$$

Therefore $a = 1$ and $b = 1$.

Now take $u = x$ and $v = x$. Then

$$f\left(\frac{2x}{1+x^2}\right) = f(x) + f(x) = 2f(x).$$

Next combine $\frac{2x}{1+x^2}$ with x:

$$\frac{\frac{2x}{1+x^2} + x}{1 + \frac{2x}{1+x^2}x} = \frac{3x + x^3}{1 + 3x^2}.$$

Therefore

$$f\left(\frac{3x + x^3}{1 + 3x^2}\right) = f\left(\frac{2x}{1+x^2}\right) + f(x) = 3f(x).$$

Hence

$$f\left(\frac{3x + x^3}{1 + 3x^2}\right) - f\left(\frac{2x}{1+x^2}\right) = 3f(x) - 2f(x) = f(x).$$

The answer is **C**.