JZ Mock Set B Paper 2

Solutions

Time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

Average difficulty: 6.975

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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Question 1

Tags: Geometry · Difficulty: 5

$PQRS$ is a rectangle.

The coordinates of P and Q are $(1, 2)$ and $(3, 6)$ respectively.

The perpendicular to PQ at Q meets the line $x + y = 10$ at R.

What is the area of $PQRS$?

A $\sqrt{5}$

B $2\sqrt{5}$

C 5

D 10

E 20

F $10\sqrt{5}$

G 50

Solution 1

Answer: D

The length of PQ is $\sqrt{(3-1)^2 + (6-2)^2} = \sqrt{20} = 2\sqrt{5}$.

The gradient of PQ is $\frac{6-2}{3-1} = 2$, so QR has gradient $-\frac{1}{2}$.

The line through $Q(3, 6)$ with gradient $-\frac{1}{2}$ is $y - 6 = -\frac{1}{2}(x - 3)$, i.e. $y = -\frac{x}{2} + \frac{15}{2}$.

Intersecting with $x + y = 10$ gives $x + \left(-\frac{x}{2} + \frac{15}{2}\right) = 10$, so $\frac{x}{2} = \frac{5}{2}$, giving $x = 5$ and $y = 5$. Thus $R = (5, 5)$.

Then $QR = \sqrt{(5-3)^2 + (5-6)^2} = \sqrt{5}$.

The area is $2\sqrt{5} \times \sqrt{5} = 10$.

Question 2

Tags: Logic Counterexample · Difficulty: 5.5

It is a fact that

$$1729 = 7 \times 13 \times 19 = 1^3 + 12^3 = 9^3 + 10^3.$$

Consider the following four statements.

(1) Every positive integer that can be written as a sum of two positive cubes in two different ways is prime.

(2) If a prime $p > 5$ can be written as $p = a^3 + b^3$ for some positive integers a, b, then p has remainder of 1 when divided by 6.

(3) No product of three distinct odd primes is one more than a perfect cube.

(4) Every positive integer of the form $a^3 + b^3$, where a and b are positive integers, has prime factors all of which are at least 3.

The fact above can be used to provide a counterexample to which of these statements?

A none of them

B 1 only

C 3 only

D 1 and 3 only

E 2 and 3 only

F 1, 2 and 3 only

G 1, 3 and 4 only

H 1, 2, 3 and 4

Solution 2

Answer: D

The given fact is

$$1729 = 7 \times 13 \times 19 = 1^3 + 12^3 = 9^3 + 10^3.$$

Statement (1) says that every positive integer which can be written as a sum of two positive cubes in two different ways is prime.

But 1729 can be written as a sum of two positive cubes in two different ways:

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Also, $1729 = 7 \times 13 \times 19$, so 1729 is not prime. Therefore the fact gives a counterexample to statement (1).

Statement (2) is about a prime $p > 5$. The number 1729 is not prime, so this fact cannot be used as a counterexample to statement (2).

Statement (3) says that no product of three distinct odd primes is one more than a perfect cube.

But

$$1729 = 7 \times 13 \times 19,$$

where 7, 13 and 19 are three distinct odd primes. Also,

$$1729 = 1^3 + 12^3 = 1 + 12^3.$$

So 1729 is one more than a perfect cube. Therefore the fact gives a counterexample to statement (3).

Statement (4) says that every positive integer of the form $a^3 + b^3$ has prime factors all of which are at least 3. For 1729, the prime factors are 7, 13 and 19, all of which are at least 3. So the fact does not give a counterexample to statement (4).

Therefore the fact can be used to provide counterexamples to statements **1 and 3 only**.

Question 3

Tags: Logic Deduction, General Algebra · Difficulty: 5.5

A student is asked to solve the equation

$$\frac{\sqrt{x+5}}{x-1} = 1.$$

The student writes:

Step (1): Multiply both sides by $(x-1)$ to obtain $\sqrt{x+5} = x-1$.

Step (2): Square both sides to obtain $x+5 = (x-1)^2 = x^2 - 2x + 1$.

Step (3): Rearrange to obtain $x^2 - 3x - 4 = 0$.

Step (4): Factorise to obtain $(x-4)(x+1) = 0$, hence $x = 4$ or $x = -1$.

The student concludes that the equation has the two solutions $x = 4$ and $x = -1$. Which one of the following statements is correct?

A Both $x = 4$ and $x = -1$ are correct solutions of the original equation.

B Neither $x = 4$ nor $x = -1$ is a correct solution of the original equation.

C Exactly one of the two values is a correct solution; the spurious value was introduced at Step (1).

D Exactly one of the two values is a correct solution; the spurious value was introduced at Step (2).

E Exactly one of the two values is a correct solution; the spurious value was introduced at Step (3).

F Exactly one of the two values is a correct solution; the spurious value was introduced at Step (4).

Solution 3

Answer: D

Test each candidate in the original equation $\frac{\sqrt{x+5}}{x-1} = 1$.

At $x = 4$: $\frac{\sqrt{9}}{3} = \frac{3}{3} = 1$. **Valid.**

At $x = -1$: $\frac{\sqrt{4}}{-2} = \frac{2}{-2} = -1 \neq 1$. **Spurious.**

So exactly one of the two values is a genuine solution; we must locate the step at which the spurious root entered.

Step (1) multiplies both sides by $(x - 1)$. This operation is reversible whenever $x \neq 1$; the only value that could possibly be introduced as an extraneous root is $x = 1$ itself. The student's polynomial does not have $x = 1$ as a root, so Step (1) introduced nothing. (The value $x = 1$ is excluded from the original by domain anyway, so even potentially this step is harmless here.)

Step (2) squares both sides of $\sqrt{x+5} = x - 1$. Squaring is not reversible: $a = b$ implies $a^2 = b^2$, but the converse fails when a and b have opposite signs. After squaring, every x with $\sqrt{x+5} = -(x-1)$ also satisfies the squared equation. At $x = -1$ we have $\sqrt{x+5} = 2$ but $x - 1 = -2$, so $x = -1$ satisfies $(\sqrt{x+5})^2 = (x-1)^2$ without satisfying $\sqrt{x+5} = x - 1$. Thus the spurious root was introduced precisely at Step (2).

Steps (3) and (4) are an algebraic rearrangement and a factorisation of a polynomial, both of which are reversible (the polynomial $x^2 - 3x - 4$ has the same root set before and after factoring), so they cannot introduce or remove roots.

The correct answer is **D**.

Question 4

Tags: — · Difficulty: 6

In a TMUA question, students are asked to determine which, if any, of the four statements 1, 2, 3 and 4 are true. The options given in the question are:

A: no statements are true

B: only statement 1 is true

C: only statement 2 is true

D: only statements 1 and 2 are true

E: only statement 3 is true

F: only statements 1 and 3 are true

G: only statement 4 is true

H: only statements 2 and 3 are true

I: only statements 2 and 4 are true

Assume that a **competent student** can always correctly determine the truth value of any statement they choose to check.

If the student is lucky, what is the **smallest number of checks** that might let them determine the correct option? Equivalently, what is the fewest checks after which it is **possible** for the student to know the correct option?

A 1

B 2

C 3

D 4

Solution 4

Answer: B

Just checking any one of the statements 1, 2, 3, 4 cannot deduce the answer even if you are lucky. For example, suppose you check statement 1 and it turns out to be true, you would still need to check 2 or 3 to determine which of D or F is the solution.

By the example above, therefore if you check 1 and it is true, then you would only need to make one more check on either 2 or 3 to deduce the answer, so the smallest number of checks possible if you are lukcy is 2, so the answer is B.

Question 5

Tags: General Number of Solutions, Graphs of Functions · Difficulty: 6

Given k is a real valued non-zero constant, and x is a non-zero real number, then

$$x^2 = \frac{k}{|x-2|}$$

has n distinct solutions. Which of the following is the **complete** list of possible values of n?

A $0,\ 2,\ 3,\ 4$

B $0,\ 1,\ 3,\ 4$

C $1,\ 2,\ 3,\ 4$

D $0,\ 1,\ 2,\ 3$

E $0,\ 2,\ 3$

F $1,\ 2,\ 3$

G $0,\ 1$

H $1,\ 2$

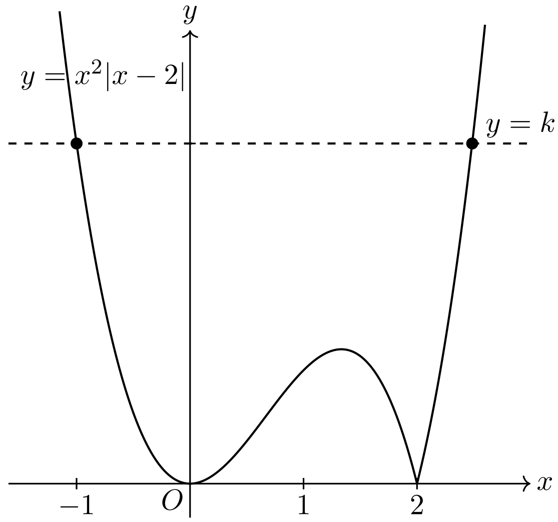
Solution 5

Answer: A

Multiply through by $|x-2| > 0$ (the equation requires $x \neq 2$):

$$x^2|x-2| = k.$$

Now consider the graph of the left and right side of the equation.



To sketch the graph of $y = x^2|x-2|$, sketch $y = x^2(x-2)$ and reflect the $x-2<0$ portion vertically.

By changing the values of k, tt is now immediately clear that achievable values of n are 0, 2, 3 and 4.

Question 6

Tags: General Algebra, Inequalities · Difficulty: 6

A list of 7 real numbers has mean 3 and range 21. The second-largest number minus the second-smallest number is 7.

What is the largest possible value of the median of the list?

A 4

B 6

C 7

D 8

E 10

F 12

Solution 6

Answer: D

method 2 logic and options guided trial and error:

First try the option 12, you will quickly realise the last 4 terms is best chosen to be $12, 12, 12, 12$, which is already over the total of $3 \times 7 = 21$, so you make the first three terms as small as possible $-9, 5, 5$ to satisfy the conditions, but together the mean is way over 3, so rule out 12.

Next try 10, you will quickly deduce best that can be done is $-11, 3, 3, 10, 10, 10, 10$ still some what over total of 21, rule out.

Now try 8, you get $-13, 1, 1, 8, 8, 8, 8$ which sum to 21, so the solution is found to be 8 and option D.

method 2 proof style, not using the given options as aid:

Order the values $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7$. The median is a_4; call it m. The constraints are

$$a_1 + a_2 + \cdots + a_7 = 21, \qquad a_7 - a_1 = 21, \qquad a_6 - a_2 = 7.$$

To maximise m, push the upper values down to m and the lower values down as far as the inequalities allow. Set $a_5 = m$ (its minimum), and let $a_2 = t$, so $a_6 = t + 7$ and $a_3 \in [t, m]$. We need $a_6 \geq m$, i.e. $t \geq m - 7$. From the range, $a_7 = a_1 + 21$, and $a_7 \geq a_6 = t + 7$ gives $a_1 \geq t - 14$, while $a_1 \leq a_2 = t$.

The sum of the six non-median entries is

$$S = a_1 + t + a_3 + m + (t + 7) + (a_1 + 21) = 2a_1 + 2t + a_3 + m + 28.$$

To allow m to be as large as possible we make S as small as possible (since $S = 21 - m$). Minimising over a_1 and a_3 gives $a_1 = t - 14$ and $a_3 = t$, so

$$S = 2(t - 14) + 2t + t + m + 28 = 5t + m.$$

Setting $S = 21 - m$: $5t + m = 21 - m$, i.e. $t = \frac{21-2m}{5}$. The binding constraint is $t \geq m - 7$:

$$\frac{21 - 2m}{5} \geq m - 7 \iff 21 - 2m \geq 5m - 35 \iff m \leq 8.$$

The maximum $m = 8$ is attained by the list -13, 1, 1, 8, 8, 8, 8, which has sum 21, range 21, and second-largest minus second-smallest equal to $8 - 1 = 7$. So the largest possible median is **8**.

Question 7

Tags: Logic Counterexample, Differentiation · Difficulty: 6.5

Consider the following statement.

(*) If $f(x)$ is an integer for every integer x, then $f'(x)$ is an integer for every integer x.

Which one of the following polynomials is a counterexample to (*)?

A $f(x) = \dfrac{x^4 + x^2 + x}{2}$

B $f(x) = \dfrac{x^3 - x}{4}$

C $f(x) = \dfrac{x^4 - x^2}{4}$

D $f(x) = \dfrac{x^2(x+1)^2}{4}$

E $f(x) = \dfrac{x^2(x+1)^2}{2}$

Solution 7

Answer: C

A counterexample requires (i) f integer-valued at every integer and (ii) f' **not** integer at some integer. We check each option.

A: $f(x) = (x^4 + x^2 + x)/2$. At $x = 1$: $f(1) = 3/2$, not an integer. The hypothesis already fails, so A is not a counterexample.

B: $f(x) = (x^3 - x)/4 = x(x-1)(x+1)/4$. At $x = 0, \pm 1$ this is 0. But at $x = 2$: $f(2) = 6/4 = 3/2$, not an integer. Hypothesis fails (at $x = \pm 2$), so B is not a counterexample. (Trap: testing only $x = 0, \pm 1$ misses the failure.)

C: $f(x) = (x^4 - x^2)/4 = x^2(x-1)(x+1)/4$. For even x, x^2 is divisible by 4. For odd x, $(x-1)$ and $(x+1)$ are consecutive even numbers, so their product is divisible by 8. Either way $f(x)$ is an integer, so the hypothesis holds. Now $f'(x) = (4x^3 - 2x)/4 = x(2x^2 - 1)/2$. At $x = 0$ this is 0 (integer), but at $x = 1$ it equals $1/2$, which is **not** an integer. So C **is** a counterexample.

D: $f(x) = x^2(x+1)^2/4 = (x(x+1)/2)^2$. Since $x(x+1)$ is even, $x(x+1)/2$ is an integer, so f is an integer (in fact a perfect square). $f'(x) = x(x+1)(2x+1)/2$, and $x(x+1)$ is even so this is always an integer. Hypothesis holds and conclusion holds, so not a counterexample.

E: $f(x) = x^2(x+1)^2/2$. Since $x(x+1)$ is even, $(x(x+1))^2$ is divisible by 4, so $f(x)$ is an even integer. $f'(x) = 2x^3 + 3x^2 + x$ has integer coefficients, hence is an integer at every integer. Not a counterexample.

Thus C is the unique counterexample.

Question 8

Tags: Sequences and Series · Difficulty: 6.5

The sequence (a_n) is defined by $a_1 = 1$ and

$$a_{n+1} = a_n + (-1)^n \cdot n \quad \text{for } n \geq 1.$$

Find the value of

$$\sum_{n=1}^{100} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + a_{99} - a_{100}.$$

A 50

B 100

C 2500

D -2500

E 2550

F 5050

G 4950

Solution 8

Answer: C

Compute the first few terms to spot the pattern:

$$\begin{aligned}
a_1 &= 1, \\
a_2 &= 1 + (-1)^1 \cdot 1 = 0, \\
a_3 &= 0 + (-1)^2 \cdot 2 = 2, \\
a_4 &= 2 + (-1)^3 \cdot 3 = -1, \\
a_5 &= -1 + (-1)^4 \cdot 4 = 3, \\
a_6 &= 3 + (-1)^5 \cdot 5 = -2, \\
a_7 &= -2 + (-1)^6 \cdot 6 = 4, \\
a_8 &= 4 + (-1)^7 \cdot 7 = -3.
\end{aligned}$$

So for $k \geq 1$,

$$a_{2k-1} = k, \qquad a_{2k} = 1 - k.$$

(Quick check: $a_{2k-1} + (-1)^{2k-1}(2k-1) = k - (2k-1) = 1 - k = a_{2k}$, and $a_{2k} + (-1)^{2k}(2k) = (1-k) + 2k = 1 + k = a_{2(k+1)-1}$. **Both transitions confirmed.**)

The required sum, grouped in pairs of two consecutive terms, becomes

$$\sum_{n=1}^{100} (-1)^{n+1} a_n = \sum_{k=1}^{50} \big(a_{2k-1} - a_{2k}\big) = \sum_{k=1}^{50} \big(k - (1-k)\big) = \sum_{k=1}^{50} (2k-1).$$

This is the sum of the first 50 odd positive integers, which equals $50^2 = 2500$.

Question 9

Tags: Logic Deduction, General Trigonometry · Difficulty: 7

The following is an attempted proof of the conjecture:

if $\sin\theta > \cos\theta$, **then** $\sin\theta > \frac{1}{\sqrt{2}}$.

Suppose $\sin\theta > \cos\theta$.

Then $\sin\theta - \cos\theta > 0$. (I)

Squaring the positive quantity $\sin\theta - \cos\theta$ gives $(\sin\theta - \cos\theta)^2 > 0$, which expands using $\sin^2\theta + \cos^2\theta = 1$ to $1 - 2\sin\theta\cos\theta > 0$, so $\sin\theta\cos\theta < \frac{1}{2}$. (II)

From (II), $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta < 2$, hence $|\sin\theta + \cos\theta| < \sqrt{2}$, so in particular $\sin\theta + \cos\theta > -\sqrt{2}$. (III)

Adding the inequalities (I) $\sin\theta - \cos\theta > 0$ and (III) $\sin\theta + \cos\theta > -\sqrt{2}$ gives $2\sin\theta > -\sqrt{2}$, so $\sin\theta > -\frac{1}{\sqrt{2}}$; squaring this inequality yields $\sin^2\theta > \frac{1}{2}$. (IV)

Since $\sin^2\theta > \frac{1}{2}$ means $|\sin\theta| > \frac{1}{\sqrt{2}}$, and $\sin\theta > -\frac{1}{\sqrt{2}}$ rules out $\sin\theta < -\frac{1}{\sqrt{2}}$, we conclude $\sin\theta > \frac{1}{\sqrt{2}}$. (V)

Which one of the following is the case?

A The proof is correct.

B The proof is incorrect, and the first error occurs in line (I).

C The proof is incorrect, and the first error occurs in line (II).

D The proof is incorrect, and the first error occurs in line (III).

E The proof is incorrect, and the first error occurs in line (IV).

F The proof is incorrect, and the first error occurs in line (V).

Solution 9

Answer: E

The conjecture is false. Counterexample: $\theta = \frac{5\pi}{6}$ gives $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$, so $\sin\theta > \cos\theta$ holds, but $\sin\theta = \frac{1}{2} < \frac{1}{\sqrt{2}}$.

Lines (I)–(III) are valid.

(I) is just a rearrangement of the hypothesis.

(II) Since $\sin\theta - \cos\theta > 0$, squaring preserves positivity: $(\sin\theta - \cos\theta)^2 > 0$. Expanding and using $\sin^2\theta + \cos^2\theta = 1$ correctly gives $1 - 2\sin\theta\cos\theta > 0$, hence $\sin\theta\cos\theta < \frac{1}{2}$.

(III) $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$, and using (II) this is less than $1 + 1 = 2$, so $|\sin\theta + \cos\theta| < \sqrt{2}$, hence $\sin\theta + \cos\theta > -\sqrt{2}$.

The error is in line (IV). Adding the inequalities is fine and correctly yields $\sin\theta > -\frac{1}{\sqrt{2}}$. The fallacy is the next step: **squaring an inequality does not preserve its direction unless both sides are non-negative**. From $\sin\theta > -\frac{1}{\sqrt{2}}$ one cannot conclude $\sin^2\theta > \frac{1}{2}$; for example $\sin\theta = 0$ satisfies $\sin\theta > -\frac{1}{\sqrt{2}}$ but $\sin^2\theta = 0 < \frac{1}{2}$. Indeed at the counterexample $\theta = \frac{5\pi}{6}$, $\sin\theta = \frac{1}{2} > -\frac{1}{\sqrt{2}}$ but $\sin^2\theta = \frac{1}{4} < \frac{1}{2}$, so (IV) fails.

Line (V) would be a valid deduction **if** (IV) were correct, so (IV) is the first error.

The answer is E.

Question 10

Tags: Logic Deduction, Integration · Difficulty: 7

Let f and g be polynomials. Consider the following four statements.

I: If $f'(x) \geq g'(x)$ for all $x \geq 0$ and $f(0) \geq g(0)$, then $f(x) \geq g(x)$ for all $x \geq 0$.

II: If $f(x) \geq g(x)$ for all $x \geq 0$, then $\int_0^x t\, f(t)\, \mathrm{d}t \geq \int_0^x t\, g(t)\, \mathrm{d}t$ for all $x \geq 0$.

III: If $\int_0^x f(t)\, \mathrm{d}t \geq \int_0^x g(t)\, \mathrm{d}t$ for all $x \geq 0$, then $f(x) \geq g(x)$ for all $x \geq 0$.

IV: If $\int_0^x f(t)\, \mathrm{d}t \geq \int_0^x g(t)\, \mathrm{d}t$ for all $x \geq 0$, then $\int_0^x t^2\, f(t)\, \mathrm{d}t \geq \int_0^x t^2\, g(t)\, \mathrm{d}t$ for all $x \geq 0$.

Which of the statements are true?

A None of them

B I only

C II only

D III only

E IV only

F I and II only

G II and III only

H I and III only

I all except I

J all except II

K all except III

L all except IV

Solution 10

Answer: F

Let $h(x) = f(x) - g(x)$. The four statements can then be considered in terms of h.

I is TRUE. If $f'(x) \geq g'(x)$ for all $x \geq 0$, then $h'(x) \geq 0$ for all $x \geq 0$. Also, $f(0) \geq g(0)$ means $h(0) \geq 0$. Since $h'(x) \geq 0$ on $x \geq 0$, h is increasing on this interval. Therefore for all $x \geq 0$, we have $h(x) \geq h(0) \geq 0$, so

$$f(x) \geq g(x).$$

II is TRUE. If $f(x) \geq g(x)$ for all $x \geq 0$, then $h(x) \geq 0$ for all $x \geq 0$. Also, for $0 \leq t \leq x$, we have $t \geq 0$. Hence $th(t) \geq 0$, so

$$\int_0^x th(t)\,\mathrm{d}t \geq 0.$$

Therefore

$$\int_0^x tf(t)\,\mathrm{d}t \geq \int_0^x tg(t)\,\mathrm{d}t.$$

III is FALSE. The condition says that the accumulated area of $f - g$ from 0 to x is always non-negative. This does not mean that $f(x) - g(x)$ itself is always non-negative.

For example, take $g(x) = 0$ and

$$f(x) = (x-1)(x-2).$$

Then

$$\int_0^x f(t)\,\mathrm{d}t = \int_0^x (t^2 - 3t + 2)\,\mathrm{d}t = \frac{x^3}{3} - \frac{3x^2}{2} + 2x.$$

This can be written as

$$\int_0^x f(t)\,\mathrm{d}t = \frac{x(2x^2 - 9x + 12)}{6}.$$

The quadratic $2x^2 - 9x + 12$ has discriminant $81 - 96 = -15$, so it is always positive. Hence

$$\int_0^x f(t)\,\mathrm{d}t \geq 0$$

for all $x \geq 0$. However, $f(x) < 0$ for $1 < x < 2$, for example $f(3/2) < 0$. So the conclusion $f(x) \geq g(x)$ for all $x \geq 0$ is false.

IV is FALSE. The condition says that

$$\int_0^x (f(t) - g(t))\,\mathrm{d}t \geq 0$$

for all $x \geq 0$. However, the conclusion requires

$$\int_0^x t^2(f(t) - g(t))\,\mathrm{d}t \geq 0$$

for all $x \geq 0$.

The factor t^2 is a non-constant weighting factor. It gives more weight to values of $f(t) - g(t)$ that occur later, so it can change the balance between the positive and negative parts.

For a counterexample, take $g(x) = 0$ and

$$f(x) = (x-1)(x-2).$$

Now $f(x) > 0$ on $0 \le x < 1$, $f(x) < 0$ on $1 < x < 2$, and $f(x) > 0$ for $x > 2$.

On $0 \le x \le 2$, the positive area from 0 to 1 is larger than the negative area from 1 to 2. In particular,

$$\int_0^2 f(t)\,\mathrm{d}t = \int_0^2 (t-1)(t-2)\,\mathrm{d}t = \frac{2}{3} > 0.$$

For $x > 2$, we are only adding more positive area, since $f(x) > 0$ for $x > 2$. Therefore

$$\int_0^x f(t)\,\mathrm{d}t \ge 0$$

for all $x \ge 0$, so the antecedent of statement IV is satisfied.

But at $x = 2$,

$$\int_0^2 t^2(f(t) - g(t))\,\mathrm{d}t = \int_0^2 t^2(t-1)(t-2)\,\mathrm{d}t.$$

Since $t^2(t-1)(t-2) = t^4 - 3t^3 + 2t^2$, we get

$$\int_0^2 t^2(f(t) - g(t))\,\mathrm{d}t = \frac{32}{5} - 12 + \frac{16}{3} = -\frac{4}{15} < 0.$$

So the weighted integral is negative, even though the ordinary accumulated integral is always non-negative. Therefore statement IV is false.

Question 11

Tags: Exponentials and Logarithms, Linear Equations · Difficulty: 7

Let p be a real constant. Consider the simultaneous equations

$$p \cdot 2^x + \log_2 y + \log_2 z = 3,$$

$$2^x + p \log_2 y + \log_2 z = 2,$$

$$2^x + \log_2 y + \log_2 z = p,$$

in real unknowns x, y, z subject to $y > 1$ and $z > 0$.

Which of the following gives the complete set of values of p for which a real solution (x, y, z) exists?

A $p > 1$

B $1 < p < 2$

C $1 < p < 3$

D $2 < p < 3$

E $p < 2$ and $p \neq 1$

F $p > 2$

G $p < 1$ or $p > 2$

H All real p except $p = 1$

Solution 11

Answer: B

Let $X = 2^x$, $Y = \log_2 y$, $Z = \log_2 z$. The constraints translate to $X > 0$, $Y > 0$ (since $y > 1$), and $Z \in \mathbb{R}$ (since $z > 0$). The system becomes

$$pX + Y + Z = 3, \quad X + pY + Z = 2, \quad X + Y + Z = p.$$

Subtract the third equation from the first: $(p-1)X = 3 - p$, so $X = \frac{3-p}{p-1}$ (requires $p \neq 1$). Subtract the third from the second: $(p-1)Y = 2 - p$, so $Y = \frac{2-p}{p-1}$. Then $Z = p - X - Y$ is automatically a real number.

Positivity of X: $\frac{3-p}{p-1} > 0$ iff $1 < p < 3$.

Positivity of Y: $\frac{2-p}{p-1} > 0$ iff $1 < p < 2$.

Intersecting: $1 < p < 2$. Check $p = 1.5$: $X = 3 > 0$, $Y = 1 > 0$, giving $x = \log_2 3$, $y = 2$, $z = 2^{1.5-3-1} = 2^{-2.5}$. Valid.

Answer: **B**.

Question 12

Tags: Logic Equivalence · Difficulty: 7

Consider the following statement about an integer n.

$$(*) \quad n^3 + 5n \text{ is divisible by } 12.$$

Use $n^3 + 5n = (n+1)n(n-1) + 6n$ or otherwise, determine which of the following is a **necessary but not sufficient** condition on n for $(*)$ to be true?

A n is 1 more than a multiple of 4.

B n is odd.

C n is even.

D n is a multiple of 3.

E n is a multiple of 4.

F n is a multiple of 6.

G n is 2 more than a multiple of 4.

Solution 12

Answer: C

A necessary condition is a consequence that must hold whenever $(*)$ is true.

Start by checking a few values of n: for $n = 1, 2, 3, 4$, the respective values of $(*)$ are $6, 18, 42, 84$. Only multiple of 12 is when $n = 4$, so this immediately rules out options A, B, D, F and G.

So the only possible answers are C and E, given E implies C, and crucially $n = 2$ case rules out C being sufficient, therefore the answer must be C, assuming the question has a unique answer. (If E were the answer, C would also qualify, contradicting uniqueness.)

Note that: it can be shown that E is in fact both necessary, and sufficient. These can be skipped, because as above, we were able to deduce the answer to the question without proving these two conditions.

Question 13

Tags: Logic Sufficiency, Logic Deduction · Difficulty: 7

Consider the six options below about a particular statement S:

A S is true if $x^2 < 1$
B S is true if and only if $x^2 < 1$
C S is true if $x^2 \leqslant 1$
D S is true if and only if $x^2 \leqslant 1$
E S is true if $x^2 < 4$
F S is true if and only if $x^2 < 4$

Given that **exactly one** of these options is correct, which one is it?

A S is true if $x^2 < 1$

B S is true if and only if $x^2 < 1$

C S is true if $x^2 \leqslant 1$

D S is true if and only if $x^2 \leqslant 1$

E S is true if $x^2 < 4$

F S is true if and only if $x^2 < 4$

Solution 13

Answer: A

Let P_1 denote $x^2 < 1$, P_2 denote $x^2 \leqslant 1$, and P_3 denote $x^2 < 4$. These are strictly nested: $P_1 \Rightarrow P_2 \Rightarrow P_3$, with each implication strict (for example, $x = 1$ satisfies P_2 but not P_1; $x = \sqrt{2}$ satisfies P_3 but not P_2).

We rule out each option in turn by checking whether being correct forces another option also to be correct (which would contradict the uniqueness assumption).

Option B: if $S \Leftrightarrow P_1$, then in particular $P_1 \Rightarrow S$, so option A would also be correct. Rejected.

Option D: if $S \Leftrightarrow P_2$, then $P_2 \Rightarrow S$ gives option C, and since $P_1 \Rightarrow P_2$ we also get $P_1 \Rightarrow S$, so option A holds too. Rejected.

Option F: if $S \Leftrightarrow P_3$, then $P_3 \Rightarrow S$ gives option E, and similarly options A and C all follow. Rejected.

Option C: if $P_2 \Rightarrow S$, then because $P_1 \Rightarrow P_2$ we deduce $P_1 \Rightarrow S$, so option A also holds. Rejected.

Option E: if $P_3 \Rightarrow S$, then $P_1 \Rightarrow P_3 \Rightarrow S$ gives option A, and $P_2 \Rightarrow P_3 \Rightarrow S$ gives option C. Rejected.

Option A (the answer): a clean witness is to let S be the statement "$-1 < x < \frac{3}{2}$". Then $P_1 \Rightarrow S$ holds, so A is correct. However, S does not imply P_1 (take $x = 1.2$), so B is incorrect. Also P_2 does not imply S (take $x = -1$), so C and D are incorrect. Finally P_3 does not imply S (take $x = -1$), so E and F are incorrect.

Thus A is internally consistent as the unique correct option, while no other option can be uniquely correct.

Question 14

Tags: Logic Negation · Difficulty: 7.5

A set S of positive integers (with $|S| \geq 3$) is called **cross-linked** if and only if for every number a in S there exist two distinct numbers b and c in S, both different from a, and a prime p, such that p divides each of a, b and c.

Let T be a set of positive integers with $|T| \geq 3$.

Which of the following statements is equivalent to T being NOT cross-linked?

A For every number a in T, and for every two distinct numbers b and c in T with $b \neq a$ and $c \neq a$, there is no prime which divides all of a, b and c.

B For every number a in T there exist two distinct numbers b and c in T, both different from a, such that no prime divides all of a, b and c.

C There exists a number a in T, and there exist two distinct numbers b and c in T with $b \neq a$ and $c \neq a$, such that no prime divides all of a, b and c.

D There exists a number a in T such that for every two distinct numbers b and c in T with $b \neq a$ and $c \neq a$, there is no prime which divides all of a, b and c.

E There exists a number a in T such that for every two distinct numbers b and c in T with $b \neq a$ and $c \neq a$, there is a prime which divides a but divides neither b nor c.

F There exists a number a in T such that for every prime p which divides a, p divides neither of any two distinct numbers b, c in T with $b \neq a$ and $c \neq a$.

G For every number a in T there exist two distinct numbers b and c in T, both different from a, and a prime p, such that p divides a but p does not divide b and p does not divide c.

H There exists a number a in T and there exists a prime p such that p divides a and for every other number b in T with $b \neq a$, p does not divide b.

Solution 14

Answer: D

Write the definition of **cross-linked** formally with quantifiers. T is cross-linked iff

$$\forall a \in T,\ \exists\, b, c \in T \text{ with } b \neq c,\ b \neq a,\ c \neq a,\ \exists \,\text{prime } p:\ (p \mid a) \wedge (p \mid b) \wedge (p \mid c).$$

Negate the whole statement, flipping each quantifier and negating the innermost conjunction:

$$\exists a \in T,\ \forall\, b, c \in T \text{ with } b \neq c,\ b \neq a,\ c \neq a,\ \forall \,\text{prime } p:\ \neg[(p \mid a) \wedge (p \mid b) \wedge (p \mid c)].$$

The inner negation says: for every prime p, it is not the case that p divides all three of a, b, c — i.e. no prime divides all three of a, b, c.

In words: there exists a number a in T such that, for every pair of distinct numbers b, c in T with $b \neq a$ and $c \neq a$, no prime divides all of a, b, c. This is option D.

Why the others fail. Option **A** flips the inner conjunction correctly but fails to flip the outer $\forall a$ to $\exists a$, so it is strictly stronger than the negation. Option **B** flips only the innermost level (no prime divides all three) but keeps both outer quantifiers as in the original; this is neither the negation nor equivalent to it. Option **C** flips the outermost $\forall a$ to $\exists a$ but keeps the next quantifier as $\exists$ (instead of flipping it to $\forall$); only one of the two outer quantifiers has been flipped. Option **E** has the correct outer quantifier pattern but mis-negates the inner statement: the negation of "$\exists p:\ (p \mid a) \wedge (p \mid b) \wedge (p \mid c)$" is "$\forall p:\ p \nmid a \vee p \nmid b \vee p \nmid c$", i.e. **no** prime divides all three; it is **not** the existence of a prime dividing a but neither b nor c. Option **F** likewise mis-negates the inner statement: it asserts that no prime factor of a divides either of b or c, which is much stronger than "no prime divides all three". Option **G** keeps the outer $\forall a$ (unflipped) and uses a wrong inner negation, so it has two errors. Option **H** collapses the two-element witness $\{b, c\}$ to a single element and asserts a prime factor of a divides no other element of T — this is the negation of **stapled** (the original 2017 Q17 answer), not of **cross-linked**.

Question 15

Tags: Integration, Inequalities · Difficulty: 7.5

Let

$$P = \int_0^1 4^{\sqrt{x}}\,dx, \qquad Q = \int_1^2 2^{\sqrt{x-1}}\,dx, \qquad R = \int_0^{1/2} 2\cdot 2^{2x}\,dx, \qquad S = \int_{-1}^1 \frac{1}{2}(\sqrt{2})^{(x+1)/2}\,dx.$$

Which of the following gives P, Q, R, S in order from **smallest to largest**?

A $S < R < Q < P$

B $P < Q < R < S$

C $S < Q < R < P$

D $R < S < Q < P$

E $S < R < P < Q$

F $Q < S < R < P$

Solution 15

Answer: A

First undo the disguised integrals by thinking about graph transformations.

The integral

$$Q = \int_1^2 2^{\sqrt{x-1}}\,dx$$

is the area under the graph of $2^{\sqrt{x}}$ on $[0,1]$, shifted one unit to the right. A horizontal shift does not change area, so

$$Q = \int_0^1 2^{\sqrt{x}}\,dx.$$

The integral

$$R = \int_0^{1/2} 2\cdot 2^{2x}\,dx$$

comes from horizontally compressing the graph of 2^x by a factor of 2, so the interval $[0,1]$ becomes $[0,\frac{1}{2}]$. This halves the area. The extra factor of 2 then doubles the height, so the original area is restored. Hence

$$R = \int_0^1 2^x\,dx.$$

The integral

$$S = \int_{-1}^{1} \frac{1}{2}(\sqrt{2})^{(x+1)/2}\,dx$$

comes from stretching the graph of $(\sqrt{2})^x$ on $[0,1]$ horizontally so that $[0,1]$ becomes $[-1,1]$. This doubles the area. The factor $\frac{1}{2}$ then halves the height, so the original area is restored. Hence

$$S = \int_0^1 (\sqrt{2})^x\,dx.$$

So the question is equivalent to comparing

$$P = \int_0^1 4^{\sqrt{x}}\,dx, \quad Q = \int_0^1 2^{\sqrt{x}}\,dx, \quad R = \int_0^1 2^x\,dx, \quad S = \int_0^1 (\sqrt{2})^x\,dx.$$

Rewrite each integrand with the common base 2:

$$4^{\sqrt{x}} = 2^{2\sqrt{x}}, \quad 2^{\sqrt{x}} = 2^{\sqrt{x}}, \quad 2^x = 2^x, \quad (\sqrt{2})^x = 2^{x/2}.$$

Since $t \mapsto 2^t$ is strictly increasing, the ordering of the four integrands on $(0,1)$ is determined by the ordering of their exponents:

$$2\sqrt{x}, \quad \sqrt{x}, \quad x, \quad \frac{x}{2}.$$

For every $x \in (0,1)$ we have $0 < x < 1$, hence $\sqrt{x} > x$ (because $\sqrt{x} - x = \sqrt{x}(1-\sqrt{x}) > 0$). Also $2\sqrt{x} > \sqrt{x}$ and $x > \frac{x}{2}$. Therefore on $(0,1)$:

$$2\sqrt{x} \; > \; \sqrt{x} \; > \; x \; > \; \frac{x}{2}.$$

Applying $2^{(\cdot)}$ preserves the strict inequalities pointwise, so on $(0,1)$:

$$4^{\sqrt{x}} \; > \; 2^{\sqrt{x}} \; > \; 2^x \; > \; (\sqrt{2})^x.$$

Integrating strict inequalities of continuous functions over $[0,1]$ gives strict inequalities of the integrals:

$$P > Q > R > S,$$

i.e. $S < R < Q < P$. The answer is **A**.

(Numerical check: $P \approx 2.649$, $Q \approx 1.608$, $R \approx 1.443$, $S \approx 1.195$.)

Question 16

Tags: General Trigonometry, Logic Sufficiency · Difficulty: 7.5

The equation

$$\sin x\,(1-\cos^2 x) = p\sin x$$

has exactly n distinct solutions in $[0, 2\pi]$, where p is a real constant.

Consider the following three statements.

I. $0 < p < \frac{1}{2}$ implies $n = 7$.

II. $n = 3$ implies $p < 0$ or $p > 1$.

III. $n = 5$ implies $p = 1$.

Which of the statements are true?

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution 16

Answer: F

Using $1-\cos^2 x = \sin^2 x$, the equation becomes

$$\sin^3 x - p\sin x = 0 \iff \sin x\,(\sin^2 x - p) = 0.$$

Thus either $\sin x = 0$ or $\sin^2 x = p$.

Roots from $\sin x = 0$ on $[0, 2\pi]$: $x \in \{0, \pi, 2\pi\}$, giving 3 solutions in every case.

Roots from $\sin^2 x = p$: If $p < 0$, there are no solutions. If $p = 0$, then $\sin x = 0$, which is identical to the first set, so no new solutions. If $0 < p < 1$, then $\sin x = \pm\sqrt{p}$ with $\sqrt{p} \in (0, 1)$; each of $\sin x = \sqrt{p}$ and $\sin x = -\sqrt{p}$ contributes 2 solutions in $[0, 2\pi]$, none coinciding with $\{0, \pi, 2\pi\}$, giving 4 new solutions. If $p = 1$, then $\sin x = \pm 1$, giving $x = \frac{\pi}{2}, \frac{3\pi}{2}$, so 2 new solutions. If $p > 1$, there are no solutions.

Combining these cases:

p	n
$p < 0$	3
$p = 0$	3
$0 < p < 1$	7
$p = 1$	5
$p > 1$	3

I. $n = 7$ if and only if $0 < p < 1$, so if $0 < p < \frac{1}{2}$ then $n = 7$. So **I** is **True.**

II. The values of p giving $n = 3$ are $p < 0$, $p = 0$ and $p > 1$. The case $p = 0$ is a counterexample to the implication $n = 3 \Rightarrow (p < 0 \text{ or } p > 1)$, because $p = 0$ satisfies neither. **False.**

III. $n = 5$ occurs only when $p = 1$, so $n = 5 \Rightarrow p = 1$. **True.**

Hence statements I and III are true, II is false. The answer is **F**.

Question 17

Tags: Logic Deduction, Sequences and Series · Difficulty: 8

In this question, $x_1, x_2, x_3, \ldots$ is an **arithmetic progression**, all of whose terms are integers.

Let n be a positive integer with $n \geq 2$. If the median of the first n terms of the sequence is an integer, which of the following four statements **must** be true?

(I) The median of $x_1, x_3, x_5, \ldots, x_{2n-1}$ is an integer.

(II) The median of the first $n+1$ terms is an integer.

(III) The median of $x_3, x_6, x_9, \ldots, x_{3n}$ is an integer.

(IV) The median of the first $4n$ terms is an integer.

A none of them

B I only

C III only

D I and II only

E I and III only

F I and IV only

G II and III only

H III and IV only

I I, II and III only

J I, III and IV only

K II, III and IV only

L I, II, III and IV

Solution 17

Answer: E

Let d denote the common difference. Since all terms are integers, $x_1, d \in \mathbb{Z}$.

Setup from the given condition. If n is odd, the median of the first n terms is $x_{(n+1)/2}$, automatically an integer; the condition places **no constraint** on d (so d may be odd or even). If n is even, the median of the first n terms is $\frac{x_{n/2}+x_{n/2+1}}{2} = x_{n/2} + \frac{d}{2}$. We are told this is an integer, therefore d must be **even** (when n is even).

Statement I. The subsequence $x_1, x_3, \ldots, x_{2n-1}$ has n terms with common difference $2d$. If n is odd, the median is the middle term x_n, an integer. If n is even, the median is $\frac{x_{n-1}+x_{n+1}}{2} = x_n$, also an integer. So I is **always true**.

Statement II. Take n odd. Then $n+1$ is even, and the median of the first $n+1$ terms is $\frac{x_{(n+1)/2}+x_{(n+3)/2}}{2} = x_{(n+1)/2} + \frac{d}{2}$. With n odd the original condition does not force d even, so this can fail. Counter-example: $x_k = k$, $n = 3$. Median of $1, 2, 3$ is 2 (integer); median of $1, 2, 3, 4$ is 2.5. So II is **not always true**.

Statement III. The subsequence $x_3, x_6, \ldots, x_{3n}$ has n terms with common difference $3d$. If n is odd, the median is the middle term $x_{3(n+1)/2}$, an integer. If n is even, the median is $\frac{x_{3n/2}+x_{3(n/2+1)}}{2} = x_{3n/2} + \frac{3d}{2}$; here n even forces d even (median of the first n terms is an integer as given in the question), so $\frac{3d}{2}$ is an integer. So III is **always true**.

Statement IV. The first $4n$ terms always form an even-length list, so the median is $\frac{x_{2n}+x_{2n+1}}{2} = x_{2n} + \frac{d}{2}$. This requires d even. With n odd, d is unconstrained, so it can fail. Counter-example: $x_k = k$, $n = 3$. Median of first 3 is 2 (integer); median of first 12 terms is $\frac{6+7}{2} = 6.5$. So IV is **not always true**.

Hence only I and III must be true. The answer is **E**.

Question 18

Tags: Logic Deduction, Differentiation · Difficulty: 8

Consider the cubic functions

$$f(x) = ax^3 + bx^2 + cx + d \quad \text{and} \quad g(x) = px^3 + qx^2 + rx + s,$$

where a, b, c, d, p, q, r, s are real constants.

It is given that $f'(x) > g'(x)$ for every real number x.

Consider the following statements:

I $a \geqslant p$

II $c > r$

III $b = q$

Which of the statements above are **necessarily** true?

A none of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution 18

Answer: E

Let $h(x) = f'(x) - g'(x)$. Then

$$h(x) = 3(a-p)x^2 + 2(b-q)x + (c-r).$$

We require $h(x) > 0$ for all real x. For a quadratic-or-lower expression $Ax^2 + Bx + C$ to be strictly positive for every real x, exactly one of the following must hold:

(Case 1) $A = 0$ and $B = 0$ and $C > 0$ (a positive constant), or

(Case 2) $A > 0$ and $B^2 - 4AC < 0$ (an upward parabola with no real roots).

Note that if $A = 0$ but $B \neq 0$, the expression is a non-constant linear function and so cannot be positive everywhere. Translating to our coefficients:

Case 1: $a = p$, $b = q$, and $c > r$.

Case 2: $a > p$, and $4(b-q)^2 < 12(a-p)(c-r)$, i.e. $(b-q)^2 < 3(a-p)(c-r)$.

We now test each statement.

Statement I ($a \geqslant p$): In Case 1 we have $a = p$, and in Case 2 we have $a > p$. So $a \geqslant p$ holds in both cases. **Necessarily true.**

Statement II ($c > r$): In Case 1, $c > r$ directly. In Case 2, $(b-q)^2 \geqslant 0$ and $a - p > 0$, so the strict inequality $(b-q)^2 < 3(a-p)(c-r)$ forces $c - r > 0$. **Necessarily true.**

Statement III ($b = q$): A counterexample is $a = 1, p = 0, b = 1, q = 0, c = 1, r = 0$. Then $h(x) = 3x^2 + 2x + 1$, with discriminant $4 - 12 = -8 < 0$, so $h(x) > 0$ for all x, yet $b \neq q$. **Not necessarily true.**

Hence the necessarily true statements are exactly I and II.

Question 19

Tags: Geometry, General Algebra · Difficulty: 8.5

A circle has centre $O = (6, 5)$. Two tangents from the point $A = (1, 0)$ touch the circle at P and Q. Given that length $AP = 2\sqrt{10}$, find the sum of the gradients of the tangent lines AP and AQ.

A $\frac{6}{3}$

B $\frac{3}{4}$

C $\frac{8}{3}$

D $\frac{3}{2}$

E $\frac{10}{3}$

F $\frac{7}{4}$

G $\frac{5}{4}$

Solution 19

Answer: E

First find the radius. Since AP is tangent at P, $\triangle APO$ has a right angle at P, so $r^2 = OA^2 - AP^2$. With $OA^2 = 5^2 + 5^2 = 50$ and $AP^2 = 40$, we get $r^2 = 10$. The circle is

$$(x-6)^2 + (y-5)^2 = 10.$$

A tangent through $A = (1, 0)$ has equation $y = m(x-1)$. Substituting into the circle:

$$(x-6)^2 + \big(m(x-1) - 5\big)^2 = 10.$$

Expanding and collecting,

$$(1+m^2)x^2 - 2(m^2 + 5m + 6)x + (m^2 + 10m + 51) = 0.$$

For tangency the discriminant vanishes, giving

$$(m^2 + 5m + 6)^2 = (1 + m^2)(m^2 + 10m + 51).$$

Expanding both sides,

$$\text{LHS} = m^4 + 10m^3 + 37m^2 + 60m + 36,$$
$$\text{RHS} = m^4 + 10m^3 + 52m^2 + 10m + 51,$$

and subtracting gives $-15m^2 + 50m - 15 = 0$, i.e.

$$3m^2 - 10m + 3 = 0.$$

Solving by the quadratic formula,

$$m = \frac{10 \pm \sqrt{100 - 36}}{6} = \frac{10 \pm 8}{6},$$

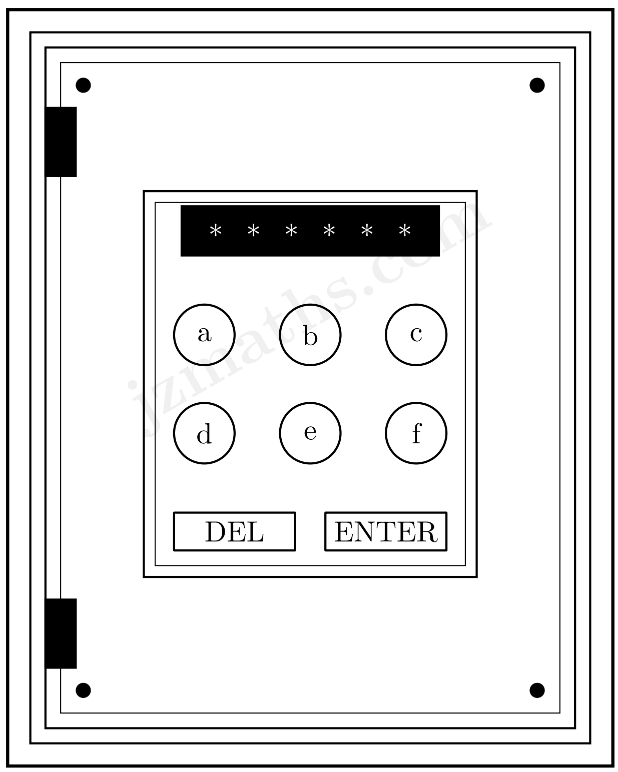
so $m = 3$ or $m = \frac{1}{3}$. The sum of the two gradients is $3 + \frac{1}{3} = \frac{10}{3}$.

The answer is **E**.

Question 20

Tags: Logic Deduction · Difficulty: 9

A safe is protected by a 6-character password that uses each of the letters a, b, c, d, e, f exactly once. After each attempt the safe reports only the number of positions in which the attempted string agrees with the password (it does not say which positions). The following three attempts have already been made: $abcdef$ gave 0 correct positions, $feadbc$ gave 0 correct positions, and $fbdcae$ gave 3 correct positions. A player now plays optimally, choosing each subsequent attempt with full knowledge of all responses received so far. What is the smallest number N such that the player is guaranteed to be able to identify the password after at most N further attempts (regardless of what the password actually is)?



A 1

B 2

C 3

D 4

E more than 5

Solution 20

Answer: B

Throughout, write a candidate password as $\pi_1\pi_2\pi_3\pi_4\pi_5\pi_6$, a permutation of $\{a, b, c, d, e, f\}$.

Step 1: Position constraints from the first two attempts.

Each attempt giving 0 correct positions tells us, for every position i, that the letter placed there is not π_i. The attempt $abcdef$ rules out $\pi_1 = a$, $\pi_2 = b$, $\pi_3 = c$, $\pi_4 = d$, $\pi_5 = e$, $\pi_6 = f$. The attempt $feadbc$ rules out $\pi_1 = f$, $\pi_2 = e$, $\pi_3 = a$, $\pi_4 = d$, $\pi_5 = b$, $\pi_6 = c$. Combining,

$$\pi_1 \notin \{a, f\}, \quad \pi_2 \notin \{b, e\}, \quad \pi_3 \notin \{a, c\}, \quad \pi_4 \neq d, \quad \pi_5 \notin \{b, e\}, \quad \pi_6 \notin \{c, f\}.$$

Step 2: Combine with the third attempt.

The attempt $fbdcae$ has 3 correct positions, so exactly 3 of the six claims

$$\pi_1 = f, \quad \pi_2 = b, \quad \pi_3 = d, \quad \pi_4 = c, \quad \pi_5 = a, \quad \pi_6 = e$$

hold. But $\pi_1 = f$ and $\pi_2 = b$ are already ruled out by Step 1, so both must be false. Exactly 3 of the remaining 4 claims must therefore hold; equivalently, exactly one of them fails.

Step 3: Enumerate the candidates.

For each choice of "false claim", the other three letters are pinned and we fill in the remaining three positions using the unused letters, subject to the Step 1 constraints.

Case A, $\pi_6 \neq e$: pin $\pi_3 = d$, $\pi_4 = c$, $\pi_5 = a$. Letters $\{b, e, f\}$ remain for positions $\{1, 2, 6\}$. The constraint $\pi_2 \notin \{b, e\}$ forces $\pi_2 = f$; then $\pi_6 \notin \{c, e, f\}$ forces $\pi_6 = b$, leaving $\pi_1 = e$. Candidate: $efdcab$.

Case B, $\pi_5 \neq a$: pin $\pi_3 = d$, $\pi_4 = c$, $\pi_6 = e$. Letters $\{a, b, f\}$ remain. Then $\pi_1 \notin \{a, f\}$ forces $\pi_1 = b$; $\pi_5 \notin \{a, b, e\}$ forces $\pi_5 = f$, leaving $\pi_2 = a$. Candidate: $badcfe$.

Case C, $\pi_4 \neq c$: pin $\pi_3 = d$, $\pi_5 = a$, $\pi_6 = e$. Letters $\{b, c, f\}$ remain for $\{1, 2, 4\}$ with $\pi_1 \in \{b, c\}$, $\pi_2 \in \{c, f\}$, $\pi_4 \in \{b, f\}$. Two valid assignments: $bcdfae$ and $cfdbae$.

Case D, $\pi_3 \neq d$: pin $\pi_4 = c$, $\pi_5 = a$, $\pi_6 = e$. Letters $\{b, d, f\}$ remain for $\{1, 2, 3\}$ with $\pi_1 \in \{b, d\}$, $\pi_2 \in \{d, f\}$, $\pi_3 \in \{b, f\}$. Two valid assignments: $bdfcae$ and $dfbcae$.

So the candidate set has six elements:

$$\{badcfe,\ bcdfae,\ bdfcae,\ cfdbae,\ dfbcae,\ efdcab\}.$$

Step 4: One further attempt cannot guarantee identification.

First observe that any two of the six remaining candidates agree in at least two positions. Now consider any one further attempt. If two candidates agree in a certain position, then that position contributes the same amount to both of their response scores: either the attempt matches both of them there, or it matches neither of them there. Since any two candidates agree in at least 2 positions, they can differ in at most 4 positions. Therefore, under any single attempt, the response scores of any two candidates can differ by at most 4.

So the range of the six possible response scores is at most 4, implies at least two of them will have to respond with the same response, and therefore it is not possible to determine the password with a single further attempt, and hence $N \geq 2$.

Step 5: Two further attempts suffice.

We now show that two further attempts are enough.

The high-level idea is to choose a test attempt that splits the six candidates into useful response groups. A good test should not be too similar to many candidates, otherwise many candidates may give the same high response. For example, a test with $bfdcae$ that gives five candidates the same response of 4 would be poor. But it should also not be too far away from all candidates, because if all candidates returned 0, we would learn nothing as well.

So the aim is to choose a test whose responses divide the candidates into manageable buckets. There are many possible attempts that can work; we only need to exhibit one valid strategy.

For attempt 4, we could for example try $abcfde$. The six candidates give responses

$$badcfe \to 1, \quad bcdfae \to 2, \quad bdfcae \to 1, \quad cfdbae \to 1, \quad dfbcae \to 1, \quad efdcab \to 0.$$

A response of 0 identifies the password as $efdcab$; a response of 2 identifies it as $bcdfae$. If the response is 1, the candidate set narrows to $\{badcfe, bdfcae, cfdbae, dfbcae\}$.

In that case, for attempt 5, try $baecdf$. The four remaining candidates give

$$badcfe \to 3, \quad bdfcae \to 2, \quad cfdbae \to 0, \quad dfbcae \to 1,$$

all distinct, so the password is identified.

Hence two further attempts always suffice, and combining with Step 4, $N = 2$. The answer is **B**.