

# JZ Mock Set B Paper 2

**Time:** 75 minutes

**Calculators:** not permitted

**Format:** 20 multiple-choice questions

**Average difficulty:** 6.975

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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**Question 1**

$PQRS$  is a rectangle.

The coordinates of  $P$  and  $Q$  are  $(1, 2)$  and  $(3, 6)$  respectively.

The perpendicular to  $PQ$  at  $Q$  meets the line  $x + y = 10$  at  $R$ .

What is the area of  $PQRS$ ?

- A  $\sqrt{5}$
- B  $2\sqrt{5}$
- C 5
- D 10
- E 20
- F  $10\sqrt{5}$
- G 50

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## Question 2

It is a fact that

$$1729 = 7 \times 13 \times 19 = 1^3 + 12^3 = 9^3 + 10^3.$$

Consider the following four statements.

- (1) Every positive integer that can be written as a sum of two positive cubes in two different ways is prime.
- (2) If a prime  $p > 5$  can be written as  $p = a^3 + b^3$  for some positive integers  $a, b$ , then  $p$  has remainder of 1 when divided by 6.
- (3) No product of three distinct odd primes is one more than a perfect cube.
- (4) Every positive integer of the form  $a^3 + b^3$ , where  $a$  and  $b$  are positive integers, has prime factors all of which are at least 3.

The fact above can be used to provide a counterexample to which of these statements?

- A none of them
- B 1 only
- C 3 only
- D 1 and 3 only
- E 2 and 3 only
- F 1, 2 and 3 only
- G 1, 3 and 4 only
- H 1, 2, 3 and 4

### Question 3

A student is asked to solve the equation

$$\frac{\sqrt{x+5}}{x-1} = 1.$$

The student writes:

**Step (1):** Multiply both sides by  $(x-1)$  to obtain  $\sqrt{x+5} = x-1$ .

**Step (2):** Square both sides to obtain  $x+5 = (x-1)^2 = x^2 - 2x + 1$ .

**Step (3):** Rearrange to obtain  $x^2 - 3x - 4 = 0$ .

**Step (4):** Factorise to obtain  $(x-4)(x+1) = 0$ , hence  $x = 4$  or  $x = -1$ .

The student concludes that the equation has the two solutions  $x = 4$  and  $x = -1$ . Which one of the following statements is correct?

- A Both  $x = 4$  and  $x = -1$  are correct solutions of the original equation.
- B Neither  $x = 4$  nor  $x = -1$  is a correct solution of the original equation.
- C Exactly one of the two values is a correct solution; the spurious value was introduced at Step (1).
- D Exactly one of the two values is a correct solution; the spurious value was introduced at Step (2).
- E Exactly one of the two values is a correct solution; the spurious value was introduced at Step (3).
- F Exactly one of the two values is a correct solution; the spurious value was introduced at Step (4).

#### Question 4

In a TMUA question, students are asked to determine which, if any, of the four statements 1, 2, 3 and 4 are true. The options given in the question are:

A: no statements are true

B: only statement 1 is true

C: only statement 2 is true

D: only statements 1 and 2 are true

E: only statement 3 is true

F: only statements 1 and 3 are true

G: only statement 4 is true

H: only statements 2 and 3 are true

I: only statements 2 and 4 are true

Assume that a **competent student** can always correctly determine the truth value of any statement they choose to check.

If the student is lucky, what is the **smallest number of checks** that might let them determine the correct option? Equivalently, what is the fewest checks after which it is **possible** for the student to know the correct option?

A 1

B 2

C 3

D 4

**Question 5**

Given  $k$  is a real valued non-zero constant, and  $x$  is a non-zero real number, then

$$x^2 = \frac{k}{|x - 2|}$$

has  $n$  distinct solutions. Which of the following is the **complete** list of possible values of  $n$ ?

**A** 0, 2, 3, 4

**B** 0, 1, 3, 4

**C** 1, 2, 3, 4

**D** 0, 1, 2, 3

**E** 0, 2, 3

**F** 1, 2, 3

**G** 0, 1

**H** 1, 2

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**Question 6**

A list of 7 real numbers has mean 3 and range 21. The second-largest number minus the second-smallest number is 7.

What is the largest possible value of the median of the list?

- A 4
- B 6
- C 7
- D 8
- E 10
- F 12

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### Question 7

Consider the following statement.

(\*) If  $f(x)$  is an integer for every integer  $x$ , then  $f'(x)$  is an integer for every integer  $x$ .

Which one of the following polynomials is a counterexample to (\*)?

**A**  $f(x) = \frac{x^4 + x^2 + x}{2}$

**B**  $f(x) = \frac{x^3 - x}{4}$

**C**  $f(x) = \frac{x^4 - x^2}{4}$

**D**  $f(x) = \frac{x^2(x+1)^2}{4}$

**E**  $f(x) = \frac{x^2(x+1)^2}{2}$

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**Question 8**

The sequence  $(a_n)$  is defined by  $a_1 = 1$  and

$$a_{n+1} = a_n + (-1)^n \cdot n \quad \text{for } n \geq 1.$$

Find the value of

$$\sum_{n=1}^{100} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + a_{99} - a_{100}.$$

- A 50
- B 100
- C 2500
- D -2500
- E 2550
- F 5050
- G 4950

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### Question 9

The following is an attempted proof of the conjecture:

**if**  $\sin \theta > \cos \theta$ , **then**  $\sin \theta > \frac{1}{\sqrt{2}}$ .

Suppose  $\sin \theta > \cos \theta$ .

Then  $\sin \theta - \cos \theta > 0$ . (I)

Squaring the positive quantity  $\sin \theta - \cos \theta$  gives  $(\sin \theta - \cos \theta)^2 > 0$ , which expands using  $\sin^2 \theta + \cos^2 \theta = 1$  to  $1 - 2\sin \theta \cos \theta > 0$ , so  $\sin \theta \cos \theta < \frac{1}{2}$ . (II)

From (II),  $(\sin \theta + \cos \theta)^2 = 1 + 2\sin \theta \cos \theta < 2$ , hence  $|\sin \theta + \cos \theta| < \sqrt{2}$ , so in particular  $\sin \theta + \cos \theta > -\sqrt{2}$ . (III)

Adding the inequalities (I)  $\sin \theta - \cos \theta > 0$  and (III)  $\sin \theta + \cos \theta > -\sqrt{2}$  gives  $2\sin \theta > -\sqrt{2}$ , so  $\sin \theta > -\frac{1}{\sqrt{2}}$ ; squaring this inequality yields  $\sin^2 \theta > \frac{1}{2}$ . (IV)

Since  $\sin^2 \theta > \frac{1}{2}$  means  $|\sin \theta| > \frac{1}{\sqrt{2}}$ , and  $\sin \theta > -\frac{1}{\sqrt{2}}$  rules out  $\sin \theta < -\frac{1}{\sqrt{2}}$ , we conclude  $\sin \theta > \frac{1}{\sqrt{2}}$ . (V)

Which one of the following is the case?

- A The proof is correct.
- B The proof is incorrect, and the first error occurs in line (I).
- C The proof is incorrect, and the first error occurs in line (II).
- D The proof is incorrect, and the first error occurs in line (III).
- E The proof is incorrect, and the first error occurs in line (IV).
- F The proof is incorrect, and the first error occurs in line (V).

**Question 10**

Let  $f$  and  $g$  be polynomials. Consider the following four statements.

**I:** If  $f'(x) \geq g'(x)$  for all  $x \geq 0$  and  $f(0) \geq g(0)$ , then  $f(x) \geq g(x)$  for all  $x \geq 0$ .

**II:** If  $f(x) \geq g(x)$  for all  $x \geq 0$ , then  $\int_0^x t f(t) dt \geq \int_0^x t g(t) dt$  for all  $x \geq 0$ .

**III:** If  $\int_0^x f(t) dt \geq \int_0^x g(t) dt$  for all  $x \geq 0$ , then  $f(x) \geq g(x)$  for all  $x \geq 0$ .

**IV:** If  $\int_0^x f(t) dt \geq \int_0^x g(t) dt$  for all  $x \geq 0$ , then  $\int_0^x t^2 f(t) dt \geq \int_0^x t^2 g(t) dt$  for all  $x \geq 0$ .

Which of the statements are true?

- A** None of them
- B** I only
- C** II only
- D** III only
- E** IV only
- F** I and II only
- G** II and III only
- H** I and III only
- I** all except I
- J** all except II
- K** all except III
- L** all except IV

**Question 11**

Let  $p$  be a real constant. Consider the simultaneous equations

$$p \cdot 2^x + \log_2 y + \log_2 z = 3,$$

$$2^x + p \log_2 y + \log_2 z = 2,$$

$$2^x + \log_2 y + \log_2 z = p,$$

in real unknowns  $x, y, z$  subject to  $y > 1$  and  $z > 0$ .

Which of the following gives the complete set of values of  $p$  for which a real solution  $(x, y, z)$  exists?

- A**  $p > 1$
- B**  $1 < p < 2$
- C**  $1 < p < 3$
- D**  $2 < p < 3$
- E**  $p < 2$  and  $p \neq 1$
- F**  $p > 2$
- G**  $p < 1$  or  $p > 2$
- H** All real  $p$  except  $p = 1$

### Question 12

Consider the following statement about an integer  $n$ .

$$(*) \quad n^3 + 5n \text{ is divisible by } 12.$$

Use  $n^3 + 5n = (n + 1)n(n - 1) + 6n$  or otherwise, determine which of the following is a **necessary but not sufficient** condition on  $n$  for  $(*)$  to be true?

- A  $n$  is 1 more than a multiple of 4.
- B  $n$  is odd.
- C  $n$  is even.
- D  $n$  is a multiple of 3.
- E  $n$  is a multiple of 4.
- F  $n$  is a multiple of 6.
- G  $n$  is 2 more than a multiple of 4.

**Question 13**

Consider the six options below about a particular statement  $S$ :

- A**  $S$  is true if  $x^2 < 1$
- B**  $S$  is true if and only if  $x^2 < 1$
- C**  $S$  is true if  $x^2 \leq 1$
- D**  $S$  is true if and only if  $x^2 \leq 1$
- E**  $S$  is true if  $x^2 < 4$
- F**  $S$  is true if and only if  $x^2 < 4$

Given that **exactly one** of these options is correct, which one is it?

- A**  $S$  is true if  $x^2 < 1$
- B**  $S$  is true if and only if  $x^2 < 1$
- C**  $S$  is true if  $x^2 \leq 1$
- D**  $S$  is true if and only if  $x^2 \leq 1$
- E**  $S$  is true if  $x^2 < 4$
- F**  $S$  is true if and only if  $x^2 < 4$

### Question 14

A set  $S$  of positive integers (with  $|S| \geq 3$ ) is called **cross-linked** if and only if for every number  $a$  in  $S$  there exist two distinct numbers  $b$  and  $c$  in  $S$ , both different from  $a$ , and a prime  $p$ , such that  $p$  divides each of  $a$ ,  $b$  and  $c$ .

Let  $T$  be a set of positive integers with  $|T| \geq 3$ .

Which of the following statements is equivalent to  $T$  being NOT cross-linked?

- A** For every number  $a$  in  $T$ , and for every two distinct numbers  $b$  and  $c$  in  $T$  with  $b \neq a$  and  $c \neq a$ , there is no prime which divides all of  $a$ ,  $b$  and  $c$ .
- B** For every number  $a$  in  $T$  there exist two distinct numbers  $b$  and  $c$  in  $T$ , both different from  $a$ , such that no prime divides all of  $a$ ,  $b$  and  $c$ .
- C** There exists a number  $a$  in  $T$ , and there exist two distinct numbers  $b$  and  $c$  in  $T$  with  $b \neq a$  and  $c \neq a$ , such that no prime divides all of  $a$ ,  $b$  and  $c$ .
- D** There exists a number  $a$  in  $T$  such that for every two distinct numbers  $b$  and  $c$  in  $T$  with  $b \neq a$  and  $c \neq a$ , there is no prime which divides all of  $a$ ,  $b$  and  $c$ .
- E** There exists a number  $a$  in  $T$  such that for every two distinct numbers  $b$  and  $c$  in  $T$  with  $b \neq a$  and  $c \neq a$ , there is a prime which divides  $a$  but divides neither  $b$  nor  $c$ .
- F** There exists a number  $a$  in  $T$  such that for every prime  $p$  which divides  $a$ ,  $p$  divides neither of any two distinct numbers  $b$ ,  $c$  in  $T$  with  $b \neq a$  and  $c \neq a$ .
- G** For every number  $a$  in  $T$  there exist two distinct numbers  $b$  and  $c$  in  $T$ , both different from  $a$ , and a prime  $p$ , such that  $p$  divides  $a$  but  $p$  does not divide  $b$  and  $p$  does not divide  $c$ .
- H** There exists a number  $a$  in  $T$  and there exists a prime  $p$  such that  $p$  divides  $a$  and for every other number  $b$  in  $T$  with  $b \neq a$ ,  $p$  does not divide  $b$ .

**Question 15**

Let

$$P = \int_0^1 4^{\sqrt{x}} dx, \quad Q = \int_1^2 2^{\sqrt{x-1}} dx, \quad R = \int_0^{1/2} 2 \cdot 2^{2x} dx, \quad S = \int_{-1}^1 \frac{1}{2} (\sqrt{2})^{(x+1)/2} dx.$$

Which of the following gives  $P$ ,  $Q$ ,  $R$ ,  $S$  in order from **smallest to largest**?

**A**  $S < R < Q < P$

**B**  $P < Q < R < S$

**C**  $S < Q < R < P$

**D**  $R < S < Q < P$

**E**  $S < R < P < Q$

**F**  $Q < S < R < P$

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**Question 16**

The equation

$$\sin x (1 - \cos^2 x) = p \sin x$$

has exactly  $n$  distinct solutions in  $[0, 2\pi]$ , where  $p$  is a real constant.

Consider the following three statements.

- I.**  $0 < p < \frac{1}{2}$  implies  $n = 7$ .
- II.**  $n = 3$  implies  $p < 0$  or  $p > 1$ .
- III.**  $n = 5$  implies  $p = 1$ .

Which of the statements are true?

- A** none of them
- B** I only
- C** II only
- D** III only
- E** I and II only
- F** I and III only
- G** II and III only
- H** I, II and III

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**Question 17**

In this question,  $x_1, x_2, x_3, \dots$  is an **arithmetic progression**, all of whose terms are integers.

Let  $n$  be a positive integer with  $n \geq 2$ . If the median of the first  $n$  terms of the sequence is an integer, which of the following four statements **must** be true?

(I) The median of  $x_1, x_3, x_5, \dots, x_{2n-1}$  is an integer.

(II) The median of the first  $n + 1$  terms is an integer.

(III) The median of  $x_3, x_6, x_9, \dots, x_{3n}$  is an integer.

(IV) The median of the first  $4n$  terms is an integer.

**A** none of them

**B** I only

**C** III only

**D** I and II only

**E** I and III only

**F** I and IV only

**G** II and III only

**H** III and IV only

**I** I, II and III only

**J** I, III and IV only

**K** II, III and IV only

**L** I, II, III and IV

### Question 18

Consider the cubic functions

$$f(x) = ax^3 + bx^2 + cx + d \quad \text{and} \quad g(x) = px^3 + qx^2 + rx + s,$$

where  $a, b, c, d, p, q, r, s$  are real constants.

It is given that  $f'(x) > g'(x)$  for every real number  $x$ .

Consider the following statements:

**I**  $a \geq p$

**II**  $c > r$

**III**  $b = q$

Which of the statements above are **necessarily** true?

**A** none of them

**B** I only

**C** II only

**D** III only

**E** I and II only

**F** I and III only

**G** II and III only

**H** I, II and III

**Question 19**

A circle has centre  $O = (6, 5)$ . Two tangents from the point  $A = (1, 0)$  touch the circle at  $P$  and  $Q$ . Given that length  $AP = 2\sqrt{10}$ , find the sum of the gradients of the tangent lines  $AP$  and  $AQ$ .

**A**  $\frac{6}{3}$

**B**  $\frac{3}{4}$

**C**  $\frac{8}{3}$

**D**  $\frac{3}{2}$

**E**  $\frac{10}{3}$

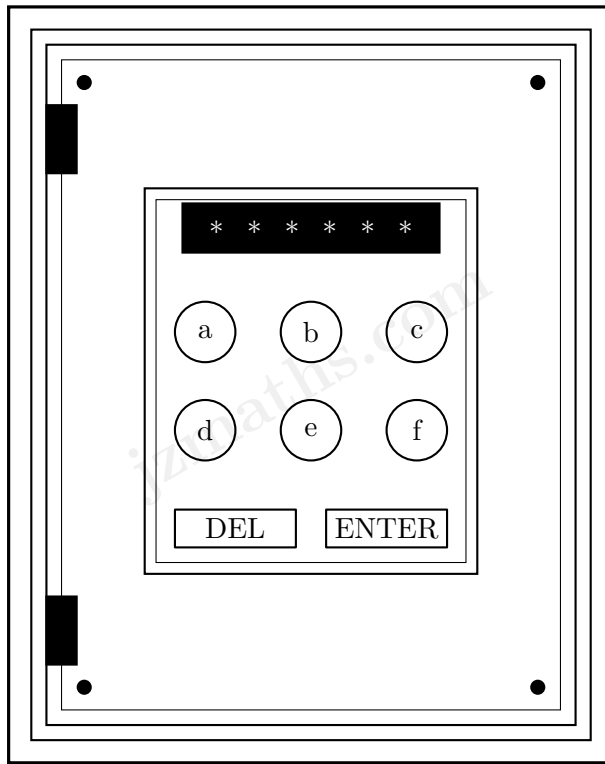
**F**  $\frac{7}{4}$

**G**  $\frac{5}{4}$

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### Question 20

A safe is protected by a 6-character password that uses each of the letters  $a, b, c, d, e, f$  exactly once. After each attempt the safe reports only the number of positions in which the attempted string agrees with the password (it does not say which positions). The following three attempts have already been made:  $abcdef$  gave 0 correct positions,  $fedabc$  gave 0 correct positions, and  $fbdcae$  gave 3 correct positions. A player now plays optimally, choosing each subsequent attempt with full knowledge of all responses received so far. What is the smallest number  $N$  such that the player is guaranteed to be able to identify the password after at most  $N$  further attempts (regardless of what the password actually is)?



- A 1
- B 2
- C 3
- D 4
- E more than 5