# JZ Mock Set B Paper 1

## Solutions

**Time:** 75 minutes

**Calculators:** not permitted

**Format:** 20 multiple-choice questions

**Average difficulty:** 6.75

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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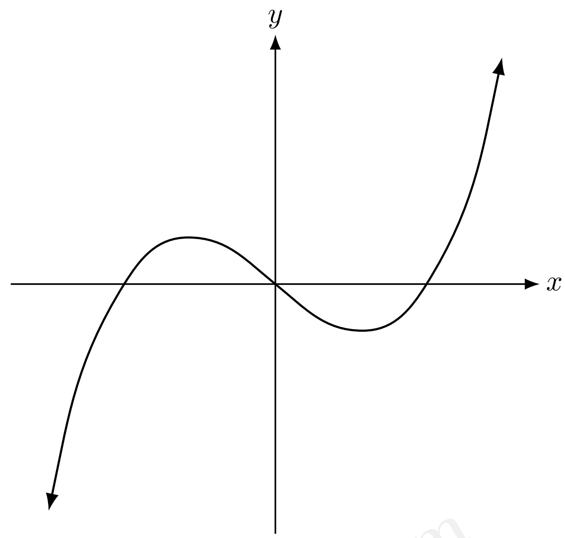
### Question 1

*Tags: Area Integration · Difficulty: 5.5*

The curve $y = x^3 - 4x$ is shown below.






What is the total area enclosed between the curve, the $x$-axis and the lines $x = -3$ and $x = 3$?

**A** $0$

**B** $\frac{9}{2}$

**C** $8$

**D** $\frac{25}{2}$

**E** $\frac{41}{2}$

**F** $\frac{33}{2}$

### Solution 1

**Answer:** E

Factorise: $y = x^3 - 4x = x(x-2)(x+2)$, so the curve crosses the $x$-axis at $x = -2, 0, 2$, all of which lie within $[-3, 3]$. The integration range therefore splits into four sub-intervals on which the curve has constant sign: on $[-3,-2]$ the curve is below the axis, on $[-2,0]$ above, on $[0,2]$ below, and on $[2,3]$ above.

Let $F(x)=\frac{x^4}{4}-2x^2$, so $F'(x)=x^3-4x$. Compute $F(-3)=\frac{81}{4}-18=\frac{9}{4}$, $F(-2)=4-8=-4$, $F(0)=0$, $F(2)=-4$, $F(3)=\frac{9}{4}$.

The four sub-areas are $|F(-2)-F(-3)|=\frac{25}{4}$, $|F(0)-F(-2)|=4$, $|F(2)-F(0)|=4$, and $|F(3)-F(2)|=\frac{25}{4}$. The total area is

$$\frac{25}{4}+4+4+\frac{25}{4}=\frac{25}{2}+8=\frac{41}{2}.$$

### Question 2

*Tags: Exponentials and Logarithms, Sequences and Series · Difficulty: 5.5*

Evaluate

$$\sum_{n=0}^{20} \log_{\frac{1}{2}}(8^{-n}).$$

**A** 630

**B** $231 \log_{\frac{1}{2}} 8$

**C** $-231 \log_{\frac{1}{2}} 8$

**D** -693

**E** $-210 \log_2 8$

**F** 693

**G** $210 \log_{\frac{1}{2}} 8$

**H** $231 \log_2 8$



### Solution 2

**Answer:** A

Pull out the constant: $\log_{\frac{1}{2}}(8^{-n}) = -n \log_{\frac{1}{2}}(8)$. Since $\log_{\frac{1}{2}}(8) = \log_{\frac{1}{2}}(2^3) = -3$, this simplifies to $3n$. Hence

$$\sum_{n=0}^{20} \log_{\frac{1}{2}}(8^{-n}) = \sum_{n=0}^{20} 3n = 3 \cdot \frac{20 \cdot 21}{2} = 630.$$

The answer is **A**.

**Question 3**

*Tags: Differentiation · Difficulty: 6*

A line is drawn normal to the curve $y = 2x^{3/2} - 4x^{1/2} - 8$ at the point on the curve where $x = 4$. This line cuts the $x$-axis at $P$ and the $y$-axis at $Q$. Find the length of $PQ$.

**A** $\frac{2\sqrt{26}}{5}$

**B** $\frac{3\sqrt{26}}{5}$

**C** $\frac{4\sqrt{26}}{5}$

**D** $\frac{2\sqrt{23}}{5}$

**E** $\frac{3\sqrt{23}}{5}$

**F** $\frac{4\sqrt{23}}{5}$



**Solution 3**

**Answer:** C

At $x = 4$,

$$y = 2(4)^{3/2} - 4(4)^{1/2} - 8 = 2(8) - 4(2) - 8 = 0.$$

So the point on the curve is $(4, 0)$.

Differentiating,

$$\frac{dy}{dx} = 3x^{1/2} - 2x^{-1/2}.$$

At $x = 4$,

$$\frac{dy}{dx} = 3(2) - 2(4)^{-1/2} = 6 - 1 = 5.$$

So the gradient of the normal is $-\frac{1}{5}$.

The normal passes through $(4, 0)$, so its equation is

$$y = -\frac{1}{5}(x - 4).$$

The line crosses the $x$-axis at $P = (4, 0)$.

To find $Q$, set $x = 0$:

$$y = -\frac{1}{5}(0 - 4) = \frac{4}{5}.$$

So $Q = (0, \frac{4}{5})$.

Therefore

$$PQ = \sqrt{(4-0)^2 + \left(0 - \frac{4}{5}\right)^2} = \sqrt{16 + \frac{16}{25}} = \sqrt{\frac{416}{25}} = \frac{4\sqrt{26}}{5}.$$

Therefore the length of $PQ$ is $\frac{4\sqrt{26}}{5}$.

## Question 4

*Tags: Circle Geometry, Transformation of Graphs · Difficulty: 6*

A circle has equation

$$x^2 + y^2 - 4x + 6y - 3 = 0.$$

The circle is first translated by 5 units in the positive $x$-direction, then reflected in the line $y = x$, then reflected in the $x$-axis, and finally enlarged by a scale factor of 3 about the origin. The equation of the final circle is

**A** $(x+9)^2 + (y+21)^2 = 144$

**B** $(x+9)^2 + (y+21)^2 = 48$

**C** $(x+9)^2 + (y+21)^2 = 16$

**D** $(x+9)^2 + (y-21)^2 = 144$

**E** $(x-9)^2 + (y+21)^2 = 144$

**F** $(x+21)^2 + (y+9)^2 = 144$

**G** $(x+3)^2 + (y+7)^2 = 144$

**H** $(x+9)^2 + (y+21)^2 = 90$



## Solution 4

**Answer:** A

Rewrite the original equation in centre–radius form by completing the square:

$$x^2 - 4x + y^2 + 6y - 3 = 0 \implies (x-2)^2 - 4 + (y+3)^2 - 9 - 3 = 0 \implies (x-2)^2 + (y+3)^2 = 16.$$

So the original centre is $(2, -3)$ and the radius is 4.

Apply the transformations to the centre (each is a rigid motion except the last). (1) Translate by $+5$ in $x$: centre becomes $(7, -3)$, radius still 4. (2) Reflect in $y = x$ (swap coordinates): centre becomes $(-3, 7)$, radius still 4. (3) Reflect in the $x$-axis (negate $y$): centre becomes $(-3, -7)$, radius still 4. (4) Enlarge by scale factor 3 about the origin (multiply each coordinate by 3, multiply radius by 3): centre becomes $(-9, -21)$, radius becomes 12.

The final equation is $(x-(-9))^2+(y-(-21))^2=12^2$, i.e. $(x+9)^2+(y+21)^2=144$.

**Question 5**

*Tags: Polynomial Expansions · Difficulty: 6*

Consider the expansion of

$$(a + bx)^n,$$

where $a$ and $b$ are positive real numbers and $n$ is a positive integer.

The second term, in **ascending** powers of $x$, is $14x$.

The third term, in **ascending** powers of $x$, is $84x^2$.

The fourth term, in **descending** powers of $x$, is $560x^4$.

Find the value of $\frac{a}{b}$.

**A** $\dfrac{1}{4}$

**B** $\dfrac{1}{2}$

**C** $\dfrac{2}{3}$

**D** $\dfrac{5}{6}$

**E** $1$



**Solution 5**

**Answer:** B

Ascending, the $(k+1)$-th term of $(a + bx)^n$ is $\binom{n}{k} a^{n-k} b^k x^k$. The fourth term descending has $x^{n-3}$, so

$$\binom{n}{3} a^3 b^{n-3} x^{n-3} = 560x^4 \implies n = 7.$$

The second ascending term is $\binom{7}{1} a^6 b\, x = 7a^6 b\, x = 14x$, giving $a^6 b = 2$.

The third ascending term is $\binom{7}{2} a^5 b^2\, x^2 = 21a^5 b^2\, x^2 = 84x^2$, giving $a^5 b^2 = 4$.

Dividing the second equation by the first: $\frac{a^5 b^2}{a^6 b} = \frac{b}{a} = \frac{4}{2} = 2$, so $\frac{a}{b} = \frac{1}{2}$.

Check: $b = 2a$ in $a^6 b = 2$ gives $a = 1$, $b = 2$, and $\binom{7}{3} a^3 b^4 = 35 \cdot 16 = 560$. ✓

The answer is **B**.

**Question 6**

*Tags: Inequalities · Difficulty: 6.5*

Let $S$ be the complete set of values of $x$ satisfying both of the following:

$$x^2 > x \qquad \text{and} \qquad 1 < 17 - 4x.$$

Which one of the following is a single inequality that represents $S$?

**A** $(x-1)(x-4) < 0$

**B** $x^2 - 5x + 4 < 0$

**C** $\dfrac{3}{x-1} > x - 3$

**D** $x(x-1)(x-4) > 0$

**E** $x(x-1) < 0$

**F** $x^2 - x > 0$



**Solution 6**

**Answer:** C

First find $S$. From $x^2 > x$: $x(x-1) > 0$, so $x < 0$ or $x > 1$. From $1 < 17 - 4x$: $x < 4$. Intersecting,

$$S = (-\infty, 0) \cup (1, 4).$$

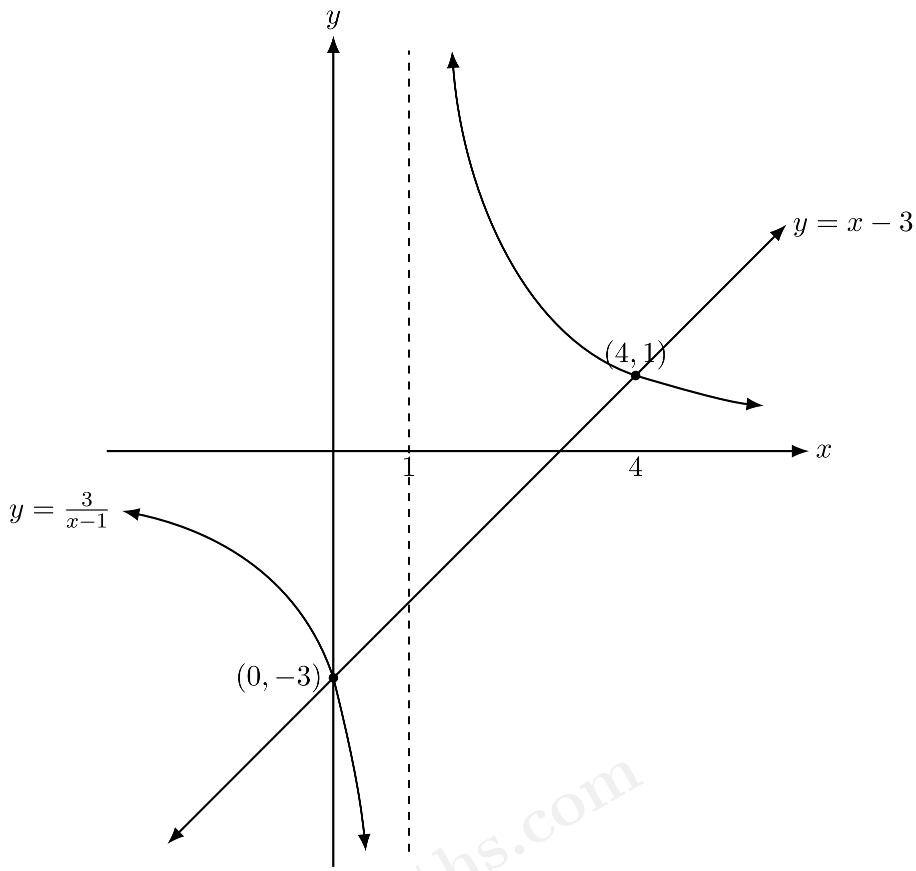
**A:** $(x-1)(x-4) < 0$ gives $1 < x < 4$, missing the $x < 0$ branch — the trap for students who treat $x^2 > x$ as $x > 1$.

**B:** $x^2 - 5x + 4 = (x-1)(x-4)$, so this is the same inequality as A.

**C:** use the graph method. Sketch $y_1 = \dfrac{3}{x-1}$ (hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = 0$) and $y_2 = x - 3$ (line through $(3, 0)$ with slope 1) on the same axes. Intersections satisfy

$$\frac{3}{x-1} = x - 3 \implies 3 = (x-1)(x-3) \implies x(x-4) = 0,$$

so the curves meet at $(0, -3)$ and $(4, 1)$.






The inequality $y_1 > y_2$ holds where the hyperbola lies above the line. The asymptote at $x = 1$ separates two such regions: $y_1 > y_2$ for $x < 0$ (failing on $0 < x < 1$ where the left branch dives to $-\infty$) and for $1 < x < 4$ (failing for $x > 4$ where the right branch decays to 0). So C represents $(-\infty, 0) \cup (1, 4) = S$.

**D:** $x(x-1)(x-4) > 0$ gives $0 < x < 1$ or $x > 4$. Wrong sign; the correct cubic representation of $S$ is $x(x-1)(x-4) < 0$.

**E:** $x(x-1) < 0$ gives $0 < x < 1$. Not $S$.

**F:** $x^2 - x = x(x-1)$ gives $x < 0$ or $x > 1$ which is also not $S$.

The answer is **C**.

## Question 7

*Tags: Exponentials and Logarithms, General Algebra, Logic Equivalence · Difficulty: 6.5*

Let $a$, $b$ and $c$ be non-zero integers. The expression

$$\frac{6^{a+b+c} \cdot 10^{a-b-c}}{15^{a-b+c}}$$

is a positive integer **if and only if** which of the following holds?

**A** $a > 0$, $b > 0$ and $c < 0$

**B** $a > 0$, $b > 0$ and $c > 0$

**C** $a > 0$, $b < 0$ and $c < 0$

**D** $a < 0$, $b > 0$ and $c < 0$

**E** $a < 0$, $b < 0$ and $c > 0$

**F** $a < 0$, $b < 0$ and $c < 0$

**G** $a > 0$ and $c < 0$ (no condition on $b$)

**H** $c < 0$ only (no conditions on $a$ or $b$)



## Solution 7

**Answer:** A

Factor each base into primes: $6 = 2 \cdot 3$, $10 = 2 \cdot 5$, $15 = 3 \cdot 5$.

The exponent of each prime in the expression is:

$$\begin{aligned} \text{prime } 2 &: (a+b+c) + (a-b-c) = 2a, \\ \text{prime } 3 &: (a+b+c) - (a-b+c) = 2b, \\ \text{prime } 5 &: (a-b-c) - (a-b+c) = -2c. \end{aligned}$$

So the expression equals $2^{2a} \cdot 3^{2b} \cdot 5^{-2c}$.

This is a positive integer iff every prime exponent is non-negative, i.e. $2a \geq 0$, $2b \geq 0$ and $-2c \geq 0$. Since $a, b, c$ are non-zero integers, this is equivalent to $a > 0$, $b > 0$ and $c < 0$.

Sanity check with $a = b = 1$, $c = -1$: numerator $= 6^1 \cdot 10^1 = 60$, denominator $= 15^{-1} = \frac{1}{15}$, so the expression equals $60 \cdot 15 = 900 = 2^2 \cdot 3^2 \cdot 5^2$, an integer, as predicted.

**Question 8**

*Tags: Inequalities · Difficulty: 6.5*

Given $-3 < x < 3$, find the total length of the intervals in which

$$\sqrt{(x-2)^2} \leq x^2 - 3x.$$

**A** $4 - \sqrt{3}$

**B** $3$

**C** $5 - \sqrt{2}$

**D** $\sqrt{2} + 1$

**E** $\sqrt{3} + 2$

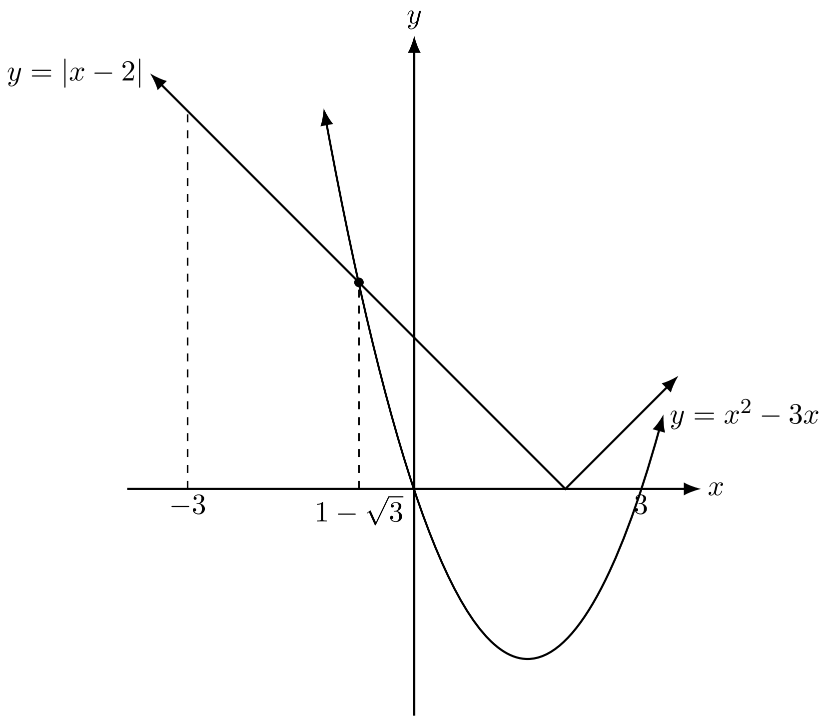
**F** $6 - 2\sqrt{3}$



**Solution 8**

**Answer:** A

Since $\sqrt{(x-2)^2} = |x-2|$, we need $|x-2| \leq x^2 - 3x$. The two graphs are shown below.




The left side is non-negative, so the inequality requires $x^2 - 3x \geq 0$, i.e., $x(x-3) \geq 0$. Combined with $-3 < x < 3$, this gives $x \in (-3, 0]$. On this range $x < 2$, so $|x - 2| = 2 - x$, and the inequality becomes

$$2 - x \leq x^2 - 3x \iff x^2 - 2x - 2 \geq 0.$$

The roots of $x^2 - 2x - 2 = 0$ are $x = 1 \pm \sqrt{3}$, so $x^2 - 2x - 2 \geq 0$ iff $x \leq 1 - \sqrt{3}$ or $x \geq 1 + \sqrt{3}$. Since $1 + \sqrt{3} > 0$, intersecting with $(-3, 0]$ leaves $x \in (-3,\ 1 - \sqrt{3}]$ — the interval in the diagram where the parabola sits on or above the V.

The total length is $(1 - \sqrt{3}) - (-3) = 4 - \sqrt{3}$. The answer is **A**.

## Question 9

*Tags: Integration · Difficulty: 6.5*

Evaluate the following integral.

$$\int_{-2}^{2} x^2|1-x^2|\,dx$$

**A** 6

**B** 2

**C** 4

**D** 8

**E** 0

## Solution 9

**Answer:** D

The integrand is even, so $\int_{-2}^{2} x^2|1-x^2|\,dx = 2\int_{0}^{2} x^2|1-x^2|\,dx$. Split at $x=1$, where $1-x^2$ changes sign.

**On** $[0,1]$**:** $|1-x^2| = 1-x^2$, so the integrand is $x^2-x^4$.

$$\int_0^1 (x^2-x^4)\,dx = \left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1 = \frac{1}{3}-\frac{1}{5} = \frac{2}{15}.$$

**On** $[1,2]$**:** $|1-x^2| = x^2-1$, so the integrand is $x^4-x^2$.

$$\int_1^2 (x^4-x^2)\,dx = \left[\frac{x^5}{5}-\frac{x^3}{3}\right]_1^2 = \frac{31}{5}-\frac{7}{3} = \frac{58}{15}.$$

Summing and doubling: $2\left(\frac{2}{15}+\frac{58}{15}\right) = 2\cdot 4 = 8$.

The answer is **D**.

## Question 10

*Tags: Polynomial Expansions, General Algebra · Difficulty: 6.5*

In the simplified expansion of $(4 + 15x)^{10}$, how many of the eleven terms have a coefficient that is divisible by 1000?

**A** 6

**B** 7

**C** 8

**D** 9

**E** 10

**F** 11

## Solution 10



**Answer:** C

The general term has coefficient

$$c_k = \binom{10}{k} 4^{10-k} 15^k = \binom{10}{k} 2^{2(10-k)} 3^k 5^k, \quad k = 0, 1, \ldots, 10.$$

We need $1000 = 2^3 \cdot 5^3$ to divide $c_k$, i.e. $v_2(c_k) \geq 3$ and $v_5(c_k) \geq 3$, where $v_p$ denotes the exponent of the prime $p$.

**Power of 2.** $v_2(c_k) = v_2\left(\binom{10}{k}\right) + 2(10 - k)$.

For $k \leq 9$ we have $2(10 - k) \geq 2$, and the only borderline case is $k = 9$ where $2(10 - k) = 2$ and $\binom{10}{9} = 10$ contributes one further factor of 2, giving $v_2 = 3$. So $v_2(c_k) \geq 3$ for all $k \leq 9$.

For $k = 10$: $c_{10} = 15^{10}$, which has $v_2 = 0$. Fails.

**Power of 5.** $v_5(c_k) = v_5\left(\binom{10}{k}\right) + k$.

For $k \geq 3$: automatic since $k \geq 3$.

For $k = 0$: $c_0 = 4^{10}$, $v_5 = 0$. Fails.

For $k = 1$: $\binom{10}{1} = 10$, so $v_5 = 1 + 1 = 2 < 3$. Fails.

For $k = 2$: $\binom{10}{2} = 45 = 3^2 \cdot 5$, so $v_5 = 1 + 2 = 3$. Passes.

**Conclusion.** Both conditions hold exactly for $k \in \{2, 3, 4, 5, 6, 7, 8, 9\}$, giving $\boxed{8}$ terms.

### Question 11

*Tags: Integration, Transformation of Graphs · Difficulty: 7*

The function $f$ is continuous for all real $x$, and

$$\int_1^5 f(x)\,dx = A.$$

Which one of the following **must** equal to $A$?

**A**

$$\int_{-2}^{2} \left[f(x+3) + x^2\right] dx$$

**B**

$$\int_{-2}^{2} \left[f(x+3) + x^3\right] dx$$

**C**

$$\int_{-2}^{2} \left[f(x-3) + x^3\right] dx$$

**D**

$$\int_{-2}^{2} \left[f(x-3) + x^2\right] dx$$

**E**

$$\int_{-1}^{3} f(x-2)dx$$

**F**

$$\int_{-1}^{3} \left[f(x+2) + x^2\right] dx$$



### Solution 11

**Answer:** B

Substitute $u = x + 3$ in

$$\int_{-2}^{2} f(x+3)\,dx$$

: then $du = dx$, and the limits transform as $x = -2 \to u = 1$, $x = 2 \to u = 5$. Hence

$$\int_{-2}^{2} f(x+3)\,dx = \int_1^5 f(u)\,du = A.$$

For the added term,

$$\int_{-2}^{2} x^3 \, dx = 0$$

since the integrand is odd and the interval is symmetric about 0. Therefore

$$\int_{-2}^{2} \left[ f(x+3) + x^3 \right] dx = A + 0 = A,$$

which gives option B.

The other options each fail in exactly one way. (A) keeps the correct $f$-shift but adds

$$\int_{-2}^{2} x^2 \, dx = \frac{16}{3}$$

instead of 0. (C) uses $f(x-3)$, which under $u = x-3$ maps $[-2,2]$ to $[-5,-1]$, not $[1,5]$. (D) compounds both errors. (E) uses $f(x-2)$, which under $u = x-2$ maps $[-1,3]$ to $[-3,1]$, not $[1,5]$. (F) has the correct shift $f(x+2)$ on $[-1,3]$, but the added

$$\int_{-1}^{3} x^2 \, dx = \frac{28}{3}$$

is non-zero since $[-1,3]$ is not symmetric about 0.

**Question 12**

*Tags: Geometry, General Algebra · Difficulty: 7*

Three sectors of circles are similar (they share the same central angle) and their radii form an arithmetic progression. The smallest sector has arc length 6. The area of the middle sector exceeds the area of the smallest sector by 21. The area of the largest sector exceeds the area of the middle sector by 27. Find the positive difference between the perimeters of the largest and smallest sectors.

**A** 7

**B** 8

**C** 10.5

**D** 12

**E** 14

**F** 16

**G** 18



**Solution 12**

**Answer:** F

Let the common central angle be $\theta$ and let the radii be $r$, $r+s$, $r+2s$ for some $s>0$. From the smallest arc, $r\theta = 6$. The area of a sector is $\frac{1}{2}r^2\theta$, so:

$\frac{1}{2}\big((r+s)^2 - r^2\big)\theta = 21 \;\Rightarrow\; (2rs+s^2)\theta = 42.$

$\frac{1}{2}\big((r+2s)^2 - (r+s)^2\big)\theta = 27 \;\Rightarrow\; (2rs+3s^2)\theta = 54.$

Subtracting gives $2s^2\theta = 12$, so $s^2\theta = 6$. Then $(2rs+s^2)\theta = 42$ becomes $2s(r\theta) + s^2\theta = 42$, i.e. $12s + 6 = 42$, so $s = 3$. Hence $\theta = \frac{6}{s^2} = \frac{2}{3}$ and $r = \frac{6}{\theta} = 9$.

The radii are 9, 12, 15 and the arc lengths are $r\theta$, $(r+s)\theta$, $(r+2s)\theta = 6, 8, 10$. The perimeter of a sector equals $2r + r\theta$, giving perimeters 24, 32, 40. The positive difference between the largest and smallest perimeters is $40 - 24 = 16$.

## Question 13

*Tags: General Trigonometry, Area Integration · Difficulty: 7*

Which of the following integrals has the **greatest** value?

You are not expected to calculate the exact values of any of these.

**A**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^{\frac{1}{3}} x)\, dx$$

**B**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos(\frac{\pi}{2} - x)|\, dx$$

**C**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^2 x)\, dx$$

**D**

$$\int_0^{\pi} \sqrt{|\sin\left(x - \frac{\pi}{2}\right)|}\, dx$$

**E**

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \cos^2 x}\, dx$$



## Solution 13

**Answer:** D

Option A is 0 since $\sin^{\frac{1}{3}} x$ is an odd function.

Option B: $|\cos(\frac{\pi}{2} - x)| = |\sin x|$, therefore value of B is 2 times the area under the graph of $\sin x$ between 0 to $\frac{\pi}{2}$.

Option E: $\sqrt{1 - \cos^2 x} = \sqrt{\sin^2 x} = |\sin x|$, so B and D have the same area, and so are both eliminated - there cannot be two correct options!

Option C: $\sin^2 x < |\sin x|$ since $-1 \leq \sin x \leq 1$, so C has an area less than B or E.

Therefore it must be D.

Let's check it anyway. $\sqrt{|\sin\left(x - \frac{\pi}{2}\right)|} = \sqrt{|\cos x|}$, therefore value of E is 2 times the area under the graph of $\sqrt{\cos x}$ between 0 to $\frac{\pi}{2}$, this is the same as that of $\sqrt{\sin x}$, which is greater than $\sin x$ since $\sqrt{a} \geq a$ if $a$ is between 0 and 1.

**Question 14**

*Tags: Sequences and Series, General Trigonometry · Difficulty: 7*

Given $a_n = \sin(\frac{2\pi}{3}n) + \cos(\frac{\pi}{6}(1+4n))$ and

$$S = \sum_{n=1}^{k} a_n.$$

For how many values of positive integer $k$ such that $1 \le k \le 100$, is $S = 0$.

**A** 0

**B** 33

**C** 34

**D** 66

**E** 67

**F** 99



**Solution 14**

**Answer:** E

Expand using the angle-sum formula:

$$\cos\frac{\pi(1+4n)}{6} = \cos\left(\frac{2\pi n}{3} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\cos\frac{2\pi n}{3} - \frac{1}{2}\sin\frac{2\pi n}{3},$$

so

$$a_n = \frac{1}{2}\sin\frac{2\pi n}{3} + \frac{\sqrt{3}}{2}\cos\frac{2\pi n}{3} = \sin\left(\frac{2\pi n}{3} + \frac{\pi}{3}\right) = \sin\frac{(2n+1)\pi}{3}.$$

This has period 3, with $a_1 = \sin\pi = 0$, $a_2 = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$, $a_3 = \sin\frac{7\pi}{3} = \frac{\sqrt{3}}{2}$. The period sum is 0, so $S_{k+3} = S_k$, and within one period

$$S_1 = 0, \quad S_2 = -\frac{\sqrt{3}}{2}, \quad S_3 = 0.$$

Hence $S_k \neq 0 \iff k \equiv 2 \pmod 3$. In $[1, 100]$ there are 33 such values (namely $2, 5, \dots, 98$), so $S_k = 0$ for the other $100 - 33 = 67$ values.

The answer is **E**.

### Question 15

*Tags: Sequences and Series, General Algebra · Difficulty: 7*

An arithmetic sequence $(a_n)$ and a convergent geometric sequence $(g_n)$ are combined to form a new sequence $(T_n)$, where $T_n = a_n + g_n$. Given that

$$T_1 = 3, \quad T_2 = 2, \quad T_3 = \frac{3}{2}, \quad T_4 = \frac{9}{8},$$

find the sum to infinity of the geometric sequence $(g_n)$.

**A** $\frac{8}{9}$

**B** $\frac{32}{27}$

**C** $\frac{4}{3}$

**D** $\frac{3}{2}$

**E** $\frac{27}{8}$

**F** 4



### Solution 15

**Answer:** B

The $n$th terms of the AP and GP are $p + (n-1)d$ and $qr^{n-1}$, so

$$T_n = p + (n-1)d + qr^{n-1}.$$

Form the first differences $\Delta_n = T_{n+1} - T_n$:

$$\begin{aligned}
\Delta_1 &= d + q(r-1) = 2 - 3 = -1, \\
\Delta_2 &= d + qr(r-1) = \frac{3}{2} - 2 = -\frac{1}{2}, \\
\Delta_3 &= d + qr^2(r-1) = \frac{9}{8} - \frac{3}{2} = -\frac{3}{8}.
\end{aligned}$$

The constant $d$ cancels in second differences:

$$\Delta_2 - \Delta_1 = q(r-1)(r-1) = q(r-1)^2 = \frac{1}{2},$$
$$\Delta_3 - \Delta_2 = q(r^2-r)(r-1) = qr(r-1)^2 = \frac{1}{8}.$$

Dividing the second by the first gives $r = \frac{1}{4}$, which satisfies $|r| < 1$ as required.

Then $q(r-1)^2 = q \cdot \frac{9}{16} = \frac{1}{2}$, so $q = \frac{8}{9}$.

Hence

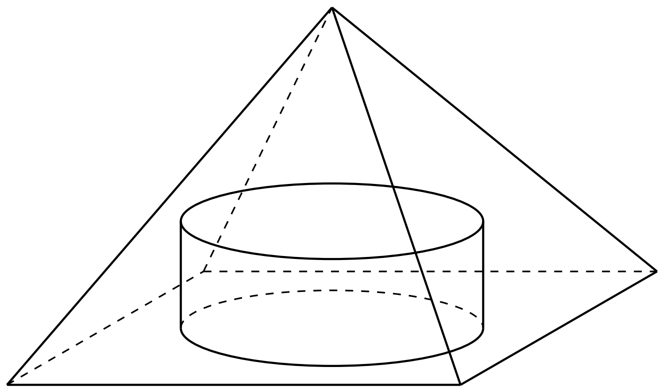
$$S_\infty = \frac{q}{1-r} = \frac{8/9}{3/4} = \frac{32}{27}.$$

(Check: $p = 3 - q = \frac{19}{9}$ and $d = -1 - q(r-1) = -\frac{1}{3}$, which reproduce all four given values of $T_n$.)

**Question 16**

*Tags: Differentiation · Difficulty: 7.5*

A regular square-based pyramid has all edges of length $L$. Find the radius of the largest-volume right circular cylinder that can be enclosed inside the pyramid.



**A** $\frac{1}{4}L$

**B** $\frac{2}{5}L$

**C** $\frac{1}{3}L$

**D** $\frac{\sqrt{2}}{5}L$

**E** $\frac{\sqrt{2}}{4}L$

**Solution 16**

**Answer:** C

**Solution**

Let the height of the pyramid be $H$. The square base has side length $L$, so the distance from the centre of the square base to one vertex is half the diagonal of the square, which is $\frac{L\sqrt{2}}{2}$. Since the sloping edge from the apex to a base vertex also has length $L$, we use Pythagoras.

$$H^2 + \left(\frac{L\sqrt{2}}{2}\right)^2 = L^2$$

$$H^2 + \frac{L^2}{2} = L^2$$

$$H^2 = \frac{L^2}{2}$$

$$H = \frac{L}{\sqrt{2}}$$

Let the cylinder have height $h$ and radius $r$. At height $h$ above the base, the horizontal cross-section of the pyramid is a smaller square similar to the base square. By similarity, the side length of this smaller square is

$$L\left(1 - \frac{h}{H}\right)$$

The circular top of the cylinder must fit inside this square. The largest circle inside a square has diameter equal to the side length of the square. Therefore,

$$2r = L\left(1 - \frac{h}{H}\right)$$

$$r = \frac{L}{2}\left(1 - \frac{h}{H}\right)$$

The volume of the cylinder is

$$V = \pi r^2 h$$

Substituting the expression for $r$ gives

$$V = \pi\left[\frac{L}{2}\left(1 - \frac{h}{H}\right)\right]^2 h$$

$$V = \frac{\pi L^2}{4} h\left(1 - \frac{h}{H}\right)^2$$

Let $u = \frac{h}{H}$. Then $h = uH$, so the volume becomes

$$V = \frac{\pi L^2 H}{4} u(1-u)^2$$

Since $\frac{\pi L^2 H}{4}$ is constant, we only need to maximise

$$u(1-u)^2$$

Let $f(u) = u(1-u)^2$. Differentiate.

$$f'(u) = (1-u)^2 - 2u(1-u)$$

$$f'(u) = (1-u)(1-3u)$$

Set $f'(u) = 0$.

$$(1-u)(1-3u) = 0$$

So

$$u = 1$$

or

$$u = \frac{1}{3}$$

When $u = 1$, the cylinder has radius 0, so its volume is 0. Therefore the maximum volume occurs when

$$u = \frac{1}{3}$$

Now substitute this into the formula for $r$.

$$r = \frac{L}{2}\left(1 - \frac{1}{3}\right)$$

$$r = \frac{L}{2} \cdot \frac{2}{3}$$

$$r = \frac{L}{3}$$

Therefore, the radius of the cylinder with maximum possible volume is $\frac{L}{3}$, and so the answer is C.

## Question 17

*Tags: General Trigonometry, Trig Equation Number of Solutions · Difficulty: 7.5*

How many solutions does the equation

$$\sqrt{1-\cos^2(2x)} = \left(x - \frac{\pi}{4}\right)\cos(2x)$$

have for $x$ in the interval $0 < x < \pi$.

**A** 0

**B** 1

**C** 2

**D** 3

**E** 4

**F** 5



## Solution 17

**Answer:** B

LHS $= |\sin(2x)| \geq 0$, so we need RHS $= (x - \pi/4)\cos(2x) \geq 0$.

**Sign analysis.** On $(0, \pi)$, $\cos(2x) > 0$ on $(0, \pi/4) \cup (3\pi/4, \pi)$ and $\cos(2x) < 0$ on $(\pi/4, 3\pi/4)$. The factor $x - \pi/4$ is negative on $(0, \pi/4)$ and positive on $(\pi/4, \pi)$. The product $(x - \pi/4)\cos(2x)$ is therefore non-negative only on $(3\pi/4, \pi)$. At $x = \pi/4, 3\pi/4$ where $\cos(2x) = 0$, the RHS is 0 but LHS $= |\sin(\pi/2)| = 1 \neq 0$, so these are not solutions.

**On** $(3\pi/4, \pi)$. Here $2x \in (3\pi/2, 2\pi)$, so $\sin(2x) < 0$ and $|\sin(2x)| = -\sin(2x)$. Dividing by $\cos(2x) > 0$ and rearranging,

$$\tan(2x) = \frac{\pi}{4} - x.$$

This asks for intersections of two graphs on $(3\pi/4, \pi)$. The curve $y = \tan(2x)$ traces a single branch of tan (over $2x \in (3\pi/2, 2\pi)$), rising from $-\infty$ to 0. The line $y = \pi/4 - x$ has slope $-1$, falling from $-\pi/2$ at $x = 3\pi/4$ to $-3\pi/4$ at $x = \pi$. A rising curve and a falling line meet at most once, and they do meet here, since the tan starts below the line ($-\infty < -\pi/2$) and ends above it ($0 > -3\pi/4$). So exactly one intersection.

The answer is **B**.

**Question 18**

*Tags: Circle Geometry, Geometry · Difficulty: 7.5*

The circles $C_1$ and $C_2$ are defined by the equations

$$C_1: \ x^2 + y^2 - 4x - 6y + 4 = 0$$

$$C_2: \ x^2 + y^2 - 22x - 30y + 330 = 0$$

A common tangent to $C_1$ and $C_2$ touches $C_1$ at $P$ and $C_2$ at $Q$. Find the sum of the squares of all distinct possible lengths of $PQ$.

**A** 1

**B** 49

**C** 176

**D** 224

**E** 400

**F** 450



**Solution 18**

**Answer:** E

**Step 1: Find centres and radii by completing the square.**

For $C_1$: $(x-2)^2 + (y-3)^2 = 4 + 9 - 4 = 9$, so centre $A = (2, 3)$ and radius $r_1 = 3$.

For $C_2$: $(x-11)^2 + (y-15)^2 = 121 + 225 - 330 = 16$, so centre $B = (11, 15)$ and radius $r_2 = 4$.

**Step 2: Distance between centres.**

$$d = |AB| = \sqrt{(11-2)^2 + (15-3)^2} = \sqrt{81 + 144} = \sqrt{225} = 15.$$

Since $d = 15 > r_1 + r_2 = 7$, the circles are disjoint, so both **external** ($PQ$ does not cross $AB$) and **internal** ($PQ$ crosses $AB$) common tangent segments exist, and these are the only two possible lengths of $PQ$.

**Step 3: External common tangent length.**

Drop a perpendicular from $A$ onto the radius $BQ$ extended; the resulting right triangle has hypotenuse $AB = d$, one leg equal to $|r_2 - r_1|$, and the other leg equal to $PQ$. Hence

$$PQ_{\text{ext}}^2 = d^2 - (r_2 - r_1)^2 = 225 - 1 = 224.$$

**Step 4: Internal common tangent length.**

For the internal case, $PQ$ crosses $AB$, and the analogous right triangle has legs $r_1 + r_2$ and $PQ$, hypotenuse $d$. Hence

$$PQ_{\text{int}}^2 = d^2 - (r_1 + r_2)^2 = 225 - 49 = 176.$$

**Step 5: Sum of squares.**

$$PQ_{\text{ext}}^2 + PQ_{\text{int}}^2 = 224 + 176 = 400.$$

**Question 19**

*Tags: General Number of Solutions, Graphs of Functions · Difficulty: 8*

Let $a$ be a real number, and let $n$ denote the number of points of intersection of the curves $y = |x^3 - a^3|$ and $y = a^3|x - 1|$. $n$ **cannot** take which of the following values?

**A** 0

**B** 1

**C** 4

**D** 3

**E** 2

**Solution 19**

**Answer:** E

Setting the curves equal: $|x^3 - a^3| = a^3|x - 1|$.



**Case** $a < 0$. Then $a^3 < 0$, so RHS $\le 0 \le$ LHS. Equality forces both sides to vanish, requiring $x = a$ and $x = 1$ simultaneously — impossible. Hence $n = 0$.

**Case** $a = 0$. The equation reduces to $|x|^3 = 0$, giving the single root $x = 0$. Hence $n = 1$.

**Case** $a > 0$. Since $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ with $x^2 + ax + a^2 > 0$, the sign of $x^3 - a^3$ matches that of $x - a$. Split on the signs of $x - a$ and $x - 1$.

In the two same-sign regions $x < \min(a, 1)$ and $x \ge \max(a, 1)$, the equation collapses to

$$x^3 = a^3 x \implies x \in \{0,\, a^{3/2},\, -a^{3/2}\}.$$

In the middle (opposite-sign) region, the equation becomes $x^3 + a^3 x - 2a^3 = 0$. The cubic on the left has derivative $3x^2 + a^3 > 0$, so it is strictly increasing with a unique real root; checking signs at $x = a$ and $x = 1$ shows this root lies strictly between them, contributing 1 intersection.

For $a = 1$ the breakpoints coincide and $|x - 1|(x^2 + x + 1) = |x - 1|$ gives $|x - 1| \cdot x(x + 1) = 0$, so $n = 3$.

For $0 < a < 1$: $a^{3/2} < a$, so all of $\{0, \pm a^{3/2}\}$ lie in the lower region (3 roots), plus the 1 middle root, giving $n = 4$.

For $a > 1$: $a^{3/2} > a$, so $\{0, -a^{3/2}\}$ lie below 1 (2 roots) and $a^{3/2}$ lies above $a$ (1 root), plus the 1 middle root, giving $n = 4$.

The achievable values of $n$ are $\{0, 1, 3, 4\}$, so $n = 2$ is impossible. The answer is **E**.

## Question 20

*Tags: Exponentials and Logarithms, General Trigonometry · Difficulty: 8*

Let $f(x) = a^{\cos x} - 2\,a^{-\cos x}$, where $a > 0$, $a \neq 1$, and $x$ is real. The difference between the maximum value of $f$ and the minimum value of $f$ is $\frac{15}{2}$.

What is the sum of all possible values of $a$?

**A** $\dfrac{5}{2}$

**B** $\dfrac{15}{4}$

**C** $\dfrac{\sqrt{41}}{2}$

**D** $\sqrt{41}$

**E** $\dfrac{5+\sqrt{41}}{4}$

**F** $\dfrac{5+\sqrt{41}}{2}$

## Solution 20

**Answer:** C

Substitute $t = a^{\cos x}$. Since $\cos x \in [-1, 1]$, the value of $t$ ranges over the closed interval with endpoints $a$ and $\frac{1}{a}$, and $t > 0$ throughout.

Write $f$ in terms of $t$: $f = t - \frac{2}{t}$. As a function of the positive variable $t$, this is **strictly increasing**, since increasing $t$ increases $t$ and decreases $-\frac{2}{t}$ (it has positive derivative $1 + \frac{2}{t^2}$, but this is not needed: monotonicity follows directly from each piece). Hence the maximum of $f$ occurs at the largest $t$ and the minimum at the smallest $t$.

**Case 1**: $a > 1$. Then $t$ ranges over $\left[\frac{1}{a}, a\right]$, so

$$\max f - \min f = \left(a - \frac{2}{a}\right) - \left(\frac{1}{a} - 2a\right) = 3a - \frac{3}{a}.$$

Setting $3a - \frac{3}{a} = \frac{15}{2}$ gives $a - \frac{1}{a} = \frac{5}{2}$, i.e. $2a^2 - 5a - 2 = 0$, so $a = \frac{5\pm\sqrt{41}}{4}$. Only $a = \frac{5+\sqrt{41}}{4}$ is positive (and exceeds 1, since $\sqrt{41} > 3$).

**Case 2**: $0 < a < 1$. Then $t$ ranges over $\left[a,\, \frac{1}{a}\right]$, so

$$\max f - \min f = \left(\frac{1}{a} - 2a\right) - \left(a - \frac{2}{a}\right) = \frac{3}{a} - 3a.$$

Setting $\frac{3}{a} - 3a = \frac{15}{2}$ gives $\frac{1}{a} - a = \frac{5}{2}$, i.e. $2a^2 + 5a - 2 = 0$, so $a = \frac{-5 \pm \sqrt{41}}{4}$. The only positive root is $a = \frac{-5+\sqrt{41}}{4}$, and since $\sqrt{41} < 7$, this is less than $\frac{1}{2} < 1$, so it is admissible.

Summing: $\frac{5+\sqrt{41}}{4} + \frac{-5+\sqrt{41}}{4} = \frac{2\sqrt{41}}{4} = \frac{\sqrt{41}}{2}$.

The answer is **C**.