JZ Mock Set A Paper 2

Solutions

Time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

Average difficulty: 6.95

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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Question 1

Tags: Logic Sufficiency, Differentiation · Difficulty: 5

Consider the equation $x^3 - 3px^2 + 4 = 0$, where p is a real parameter.

Which of the following is a **sufficient** but **not necessary** condition on p for this equation to have exactly one real root?

A $p > 1$

B $p \geq 1$

C $p \leq 1$

D $|p| \geq 1$

E $p^2 > 1$

F $-1 < p < 1$

G $p \in \mathbb{R}$

H $p < 1$

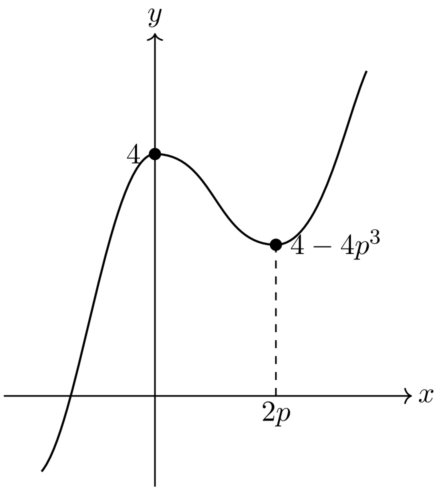
Solution 1

Answer: F

Let $f(x) = x^3 - 3px^2 + 4$. Then $f'(x) = 3x^2 - 6px = 3x(x - 2p)$, with stationary points at $x = 0$ and $x = 2p$, and stationary values

$$f(0) = 4, \qquad f(2p) = 8p^3 - 12p^3 + 4 = 4 - 4p^3.$$

For a cubic with positive leading coefficient, exactly one real root occurs iff both stationary values are strictly positive (or both strictly negative). Since $f(0) = 4 > 0$ here, the condition reduces to $f(2p) > 0$, i.e. $4 - 4p^3 > 0 \iff p < 1$. The configuration is shown below.



So $p < 1$ is both **sufficient and necessary**. The interval $-1 < p < 1$ is a proper subset of $p < 1$, hence sufficient but not necessary.

The answer is **F**.

Question 2

Tags: Differentiation · Difficulty: 5

What is the gradient of the curve

$$y = \frac{(\sqrt{x}+2)^3}{x\sqrt{x}}$$

at the point where $x = 4$?

A $-\dfrac{5}{2}$

B $-\dfrac{3}{2}$

C $-\dfrac{7}{2}$

D $\dfrac{7}{2}$

E $\dfrac{5}{2}$

F $\dfrac{3}{2}$

Solution 2

Answer: B

Expand the numerator using the binomial theorem with $u = \sqrt{x}$:

$$(\sqrt{x}+2)^3 = x^{3/2} + 6x + 12\sqrt{x} + 8.$$

Divide each term by $x\sqrt{x} = x^{3/2}$ to put y in power form:

$$y = 1 + 6x^{-1/2} + 12x^{-1} + 8x^{-3/2}.$$

Differentiate term by term:

$$\frac{dy}{dx} = -3x^{-3/2} - 12x^{-2} - 12x^{-5/2}.$$

At $x = 4$ we have $4^{-3/2} = \frac{1}{8}$, $4^{-2} = \frac{1}{16}$, $4^{-5/2} = \frac{1}{32}$, so

$$\frac{dy}{dx} = -\frac{3}{8} - \frac{3}{4} - \frac{3}{8} = -\frac{3}{2}.$$

Question 3

Tags: Logic Deduction, Logic Counterexample · Difficulty: 5.5

A student attempts to prove the following statement.

Claim: Consider integers a and b, where a has remainder 1 when divided by 3, and b has remainder 2 when divided by 3. Then $a^2 + b^2 + 1$ is always divisible by 6.

Consider the following attempt:

$$\text{Let } a = 3n + 1 \text{ and } b = 3n + 2 \qquad \text{(I)}$$

$$\text{then } a^2 + b^2 + 1 = (3n + 1)^2 + (3n + 2)^2 + 1 \qquad \text{(II)}$$

$$\text{so } a^2 + b^2 + 1 = 9n^2 + 6n + 1 + 9n^2 + 12n + 4 + 1 \qquad \text{(III)}$$

$$\text{so } a^2 + b^2 + 1 = 18n^2 + 18n + 6 = 6(3n^2 + 3n + 1) \qquad \text{(IV)}$$

$$\text{therefore } a^2 + b^2 + 1 \text{ is always divisible by 6.} \qquad \text{(V)}$$

Which of the following best describes this proof?

A The statement is true and the proof is completely correct.

B The statement is true but there is an error in the proof in line (I).

C The statement is true but there is an error in the proof in line (III).

D The statement is not true and the error first occurs in line (I).

E The statement is not true and the error first occurs in line (IV).

F The statement is not true and the error first occurs in line (V).

Solution 3

Answer: D

Locating the error. In line (I) the student writes both a and b in terms of the **same** integer n. The hypothesis only tells us $a \equiv 1 \pmod 3$ and $b \equiv 2 \pmod 3$; these remainders are independent, so a and b should be parameterised by **different** integers, e.g. $a = 3m + 1$ and $b = 3n + 2$. The error therefore first occurs in line (I).

Testing the claim. With independent variables,

$$a^2+b^2+1=(3m+1)^2+(3n+2)^2+1=9m^2+9n^2+6m+12n+6=3(3m^2+3n^2+2m+4n+2).$$

This is divisible by 3, but not necessarily by 6 — divisibility by 6 requires $3m^2+3n^2+2m+4n+2$ to be even.

Take $m=1$, $n=0$: then $a=4$, $b=2$, and $a^2+b^2+1=16+4+1=21$, which is **not** divisible by 6. So the claim is false.

(The flawed proof appears to work because forcing a and b to share the parameter n makes $3m^2+3n^2+2m+4n+2$ collapse to $6n^2+6n+2$, which is always even — masking the failure for general m, n.)

Hence the statement is not true and the first error occurs in line (I): **D**.

Question 4

Tags: Logic Counterexample, General Algebra · Difficulty: 5.5

Consider the following statement:

If a positive integer N has the property that N^2 is divisible by 12, then N is divisible by 12.

Which of the following are counterexamples to this statement?

I $N = 6$

II $N = 8$

III $N = 18$

IV $N = 24$

A none of them

B I only

C II only

D I and III only

E I, II and III only

F II and IV only

G I, III and IV only

H I, II, III and IV

Solution 4

Answer: D

A counterexample to "if P then Q" is a value for which P holds but Q fails. Here P is "N^2 is divisible by 12" and Q is "N is divisible by 12".

First, characterise when $12 \mid N^2$. Since $12 = 2^2 \cdot 3$, we need $4 \mid N^2$ and $3 \mid N^2$. As 3 is prime, $3 \mid N^2 \Rightarrow 3 \mid N$. Writing $N = 2^a m$ with m odd-times-something, $4 \mid N^2 = 2^{2a}m^2$ requires $2a \geq 2$, i.e. $a \geq 1$, so $2 \mid N$. Hence $12 \mid N^2 \iff 6 \mid N$.

So P holds precisely when $6 \mid N$, and a counterexample needs $6 \mid N$ but $12 \nmid N$.

I: $N = 6$. $N^2 = 36 = 12 \cdot 3$, so P holds. $12 \nmid 6$, so Q fails. **Counterexample.**

II: $N = 8$. $N^2 = 64$, and $64 = 12 \cdot 5 + 4$, so $12 \nmid 64$. The premise P already fails, so this is **not** a counterexample (an implication is not refuted by a case where P is false; it is only tested when P holds).

III: $N = 18$. $N^2 = 324 = 12 \cdot 27$, so P holds. $12 \nmid 18$, so Q fails. **Counterexample.**

IV: $N = 24$. $N^2 = 576 = 12 \cdot 48$, so P holds. But $24 = 12 \cdot 2$, so Q also holds. The implication is satisfied here, so this is **not** a counterexample.

Only I and III are counterexamples.

Question 5

Tags: Sequences and Series · Difficulty: 6

Sequence 1 is an arithmetic progression with first term 10 and common difference 4.

Sequence 2 is an arithmetic progression with first term 8 and common difference 6.

Sequence 3 is an arithmetic progression with first term 23 and common difference 5.

Some positive integers appear in all three of Sequence 1, Sequence 2 and Sequence 3. Let N be the 17th such integer.

What is the remainder when N is divided by 7?

A 0

B 1

C 2

D 3

E 4

F 5

G 6

Solution 5

Answer: E

The numbers common to all three sequences form an arithmetic progression with common difference $\operatorname{lcm}(4,6,5) = 60$, since stepping from one common term to the next requires a step that is a multiple of all three step sizes.

Find the first common term by intersecting in stages. Sequences 1 and 2 first meet at 14 $(= 10+4 = 8+6)$; thereafter they share an AP with common difference $\operatorname{lcm}(4,6) = 12$, giving $14, 26, 38, 50, \ldots$. Comparing with Sequence 3 $(23, 28, 33, 38, \ldots)$, the smallest match is 38.

Hence the common terms form an AP with first term 38 and common difference 60. (Check: $38 = 10 + 4 \cdot 7 = 8 + 6 \cdot 5 = 23 + 5 \cdot 3$.)

The 17th term is

$$N = 38 + 16 \cdot 60 = 998.$$

Since $998 = 7 \cdot 142 + 4$, the remainder when N is divided by 7 is 4. The answer is E.

Question 6

Tags: Logic Deduction, Inequalities · Difficulty: 6

The real numbers a, b, c and d satisfy $a \leqslant b$, with a and b both non-zero, and $c < d$.

Which of the following statements are **necessarily** true?

I $\dfrac{1}{a^3} \geq \dfrac{1}{b^3}$

II $3^{-a} \geqslant 3^{-b}$

III $a(d-c) \leqslant b(d-c)$

IV $\dfrac{a}{c^2+1} \leqslant \dfrac{b}{d^2+1}$

A none of them

B **II** only

C **III** only

D **II** and **III** only

E **I**, **II** and **III** only

F **II**, **III** and **IV** only

G **I**, **II** and **IV** only

H **I**, **II**, **III** and **IV**

Solution 6

Answer: D

We test each statement:

I: $\frac{1}{a^3} \geqslant \frac{1}{b^3}$. Take $a = -1$, $b = 1$ (allowed: both non-zero, $a \leqslant b$). Then $\frac{1}{a^3} = -1$ and $\frac{1}{b^3} = 1$, so $\frac{1}{a^3} < \frac{1}{b^3}$. The cube preserves sign, so the reciprocal of a negative cube is negative and the reciprocal of a positive cube is positive — the inequality reverses only when both sides have the same sign. Hence **I** is not necessarily true.

II: $3^{-a} \geqslant 3^{-b}$. The function $x \mapsto 3^{-x} = (1/3)^x$ is strictly decreasing, so $a \leqslant b \Rightarrow 3^{-a} \geqslant 3^{-b}$. Always true.

III: $a(d-c) \leqslant b(d-c)$. Since $c < d$, we have $d - c > 0$. Multiplying $a \leqslant b$ by the strictly positive number $d - c$ preserves the inequality. Always true.

IV: $\frac{a}{c^2+1} \leqslant \frac{b}{d^2+1}$. Both denominators are positive, but they are different, so we cannot just compare numerators. Take $a = b = 1$ (allowed: $a \leqslant b$, both non-zero), $c = 0$, $d = 1$. Then LHS $= \frac{1}{1} = 1$ and RHS $= \frac{1}{2}$, so LHS $>$ RHS. Not necessarily true.

Only **II** and **III** are necessarily true. Answer: **D**.

Question 7

Tags: Logic Deduction · Difficulty: 6.5

A student attempts to prove the following claim. **Claim:** For every positive integer n, if $n^2 + 2$ is prime, then n is a multiple of 3. The student's argument is given line by line below.

I. Suppose n is a positive integer such that $n^2 + 2$ is prime, but n is not a multiple of 3.
II. Then the remainder of n on division by 3 is 1 or 2.
III. In either case, n^2 leaves remainder 1 when divided by 3.
IV. Therefore $n^2 + 2$ leaves remainder 0 when divided by 3.
V. So 3 divides $n^2 + 2$.
VI. Hence $n^2 + 2 = 3k$ for some integer k.
VII. It follows that $n^2 + 2$ is composite and not prime.
VIII. This contradicts the assumption in (I) that $n^2 + 2$ is prime.
IX. Therefore n is a multiple of 3.

Which one of the following statements about the argument is correct?

A The argument is completely correct.

B The first error in the argument is on line I.

C The first error in the argument is on line II.

D The first error in the argument is on line III.

E The first error in the argument is on line IV.

F The first error in the argument is on line V.

G The first error in the argument is on line VI.

H The first error in the argument is on line VII.

I The first error in the argument is on line VIII.

J The first error in the argument is on line IX.

Solution 7

Answer: H

First, the claim is **false**: take $n = 1$, giving $n^2 + 2 = 3$, which is prime, yet 1 is not a multiple of 3. So the proof must contain an error somewhere; we need the **first** line that fails.

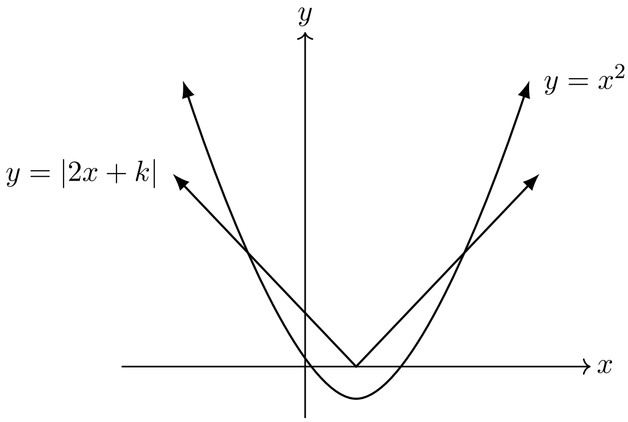
Lines I to V are correct. Line I sets up the contradiction. Line II uses the standard fact that every integer leaves remainder 0, 1 or 2 on division by 3, and (I) rules out 0. Line III: if $n = 3m + 1$ then $n^2 = 9m^2 + 6m + 1$, remainder 1; if $n = 3m + 2$ then $n^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$, remainder 1 again. Line IV adds 2 to a remainder of 1 to get 3, i.e. remainder 0. Line V is just (IV) restated in divisibility language.

Line VII is where the error occurs, as we would need to check if $3k$ could be prime for some integer value of k, and it can be, for $k = 1$ and $n = 1$. Therefore line VII is not true in general so the answer is H.

Question 8

Tags: Inequalities · Difficulty: 7

For some values of k, the graphs of $y = |2x + k|$ and $y = x^2 + (k - 1)x + 5$ intersect, as in the diagram below.



Find the complete set of values of k for which the graph $y = |2x + k|$ lies strictly below the curve $y = x^2 + (k - 1)x + 5$ for every real value of x.

A $1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}$

B $k < 1 + 2\sqrt{7}$

C $1 - 2\sqrt{5} < k < 1 + 2\sqrt{5}$

D $1 - 2\sqrt{3} < k < 1 + 2\sqrt{5}$

E $k < 1 - 2\sqrt{3}$ or $k > 1 + 2\sqrt{3}$

F There are no such values of k.

G $k < 1 - 2\sqrt{5}$ or $k > 1 + 2\sqrt{5}$

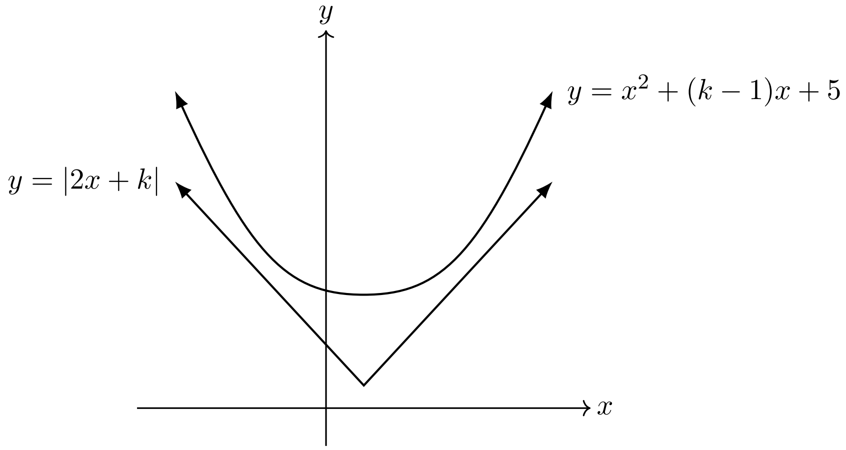
Solution 8

Answer: A

The line $y = |2x + k|$ lies strictly below the curve iff

$$|2x + k| < x^2 + (k-1)x + 5 \quad \text{for all } x \in \mathbb{R},$$

which splits into two simultaneous conditions: $2x + k < x^2 + (k-1)x + 5$ AND $-(2x + k) < x^2 + (k-1)x + 5$, i.e. the parabola must dominate both branches of the V.



Rearranging gives

$$x^2 + (k-3)x + (5-k) > 0 \quad \text{and} \quad x^2 + (k+1)x + (5+k) > 0$$

for all x. Each has positive leading coefficient, so each holds everywhere iff its discriminant is strictly negative.

First discriminant:

$$(k-3)^2 - 4(5-k) = k^2 - 2k - 11 < 0 \implies 1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}.$$

Second discriminant:

$$(k+1)^2 - 4(5+k) = k^2 - 2k - 19 < 0 \implies 1 - 2\sqrt{5} < k < 1 + 2\sqrt{5}.$$

Intersecting (the first interval is tighter, since $2\sqrt{3} < 2\sqrt{5}$):

$$1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}.$$

The inequalities are strict because **strictly below** excludes tangency.

Question 9

Tags: Logic Deduction, Inequalities · Difficulty: 7

The region R in the (x, y)-plane consists of all points satisfying **both**

$$|y - x^2| < 3 \qquad \text{and} \qquad x + y < 4.$$

Consider the following three claims about points in R.

I: For every $(x, y) \in R$, $y < 7$.

II: For every $(x, y) \in R$, $y > -3$.

III: For every $(x, y) \in R$, $x < 4$.

Which of the claims are true?

A None of them

B I only

C II only

D III only

E I and II only

F I and III only

G II and III only

H I, II and III

Solution 9

Answer: G

Rewrite the defining conditions of R as

$$x^2 - 3 < y < x^2 + 3 \qquad \text{and} \qquad y < 4 - x.$$

Claim II is true. The first condition gives $y > x^2 - 3$. Since $x^2 \geq 0$, this yields $y > -3$ for every point of R.

Claim III is true. Combining $y > x^2 - 3$ with $y < 4 - x$ forces

$$x^2 - 3 < 4 - x \quad \iff \quad x^2 + x - 7 < 0.$$

The roots of $x^2 + x - 7 = 0$ are $x = \frac{-1 \pm \sqrt{29}}{2}$, so for any point of R,

$$x < \frac{-1 + \sqrt{29}}{2} \approx 2.19,$$

which is well below 4. Hence $x < 4$ holds throughout R.

Claim I is false. We must produce a point of R with $y \geq 7$. Try $x = -3.1$. Then $x^2 = 9.61$, so the band $|y - x^2| < 3$ becomes $6.61 < y < 12.61$; and $x + y < 4$ becomes $y < 7.1$.

Intersecting, y may take any value in $(6.61, 7.1)$. Choose $y = 7$. Then

$$|7 - 9.61| = 2.61 < 3 \quad \text{and} \quad -3.1 + 7 = 3.9 < 4,$$

so $(-3.1, 7) \in R$, yet $y = 7$ is not less than 7. (Indeed any $y \in (7, 7.1)$ such as $y = 7.05$ also works and gives a strict counterexample.) So I fails.

Therefore exactly II and III hold, giving answer **G**.

Question 10

Tags: Integration, Logic Equivalence · Difficulty: 7

Let k be a real number, and consider the following two statements.

R: "k is an integer multiple of π".

S: "$\int_0^k \left(\sin x + \sin^3(2x)\right)\, dx = 0$".

Which of the following best describes the logical relationship between R and S?

A R is necessary and sufficient for S.

B R is necessary but not sufficient for S.

C R is sufficient but not necessary for S.

D R is neither necessary nor sufficient for S.

Solution 10

Answer: B

Read S graphically as: the signed area under $f(x) = \sin x + \sin 2x$ from 0 to k is zero. Let $F(k)$ denote this signed area and sketch f on $[0, 2\pi]$.

Net area over a full period. Both $\sin x$ and $\sin 2x$ have equal positive and negative bumps on $[0, 2\pi]$ ($\sin 2x$ completes two periods there), so f has zero net signed area over any $[2n\pi, 2(n+1)\pi]$. Hence F is 2π-periodic with $F(2n\pi) = 0$, and every $k = 2n\pi$ satisfies S.

Net area over $[0, \pi]$. On this interval $\sin 2x$ still has equal positive and negative bumps (net zero), but $\sin x \geq 0$ contributes strictly positive area. So $F(\pi) > 0$, and in particular $k = \pi$ does not satisfy S.

No other zeros in $(0, 2\pi)$. Factoring $f(x) = \sin x(1 + 2\cos x)$, the zeros of f on $[0, 2\pi]$ are $0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$, with f positive on $(0, \frac{2\pi}{3}) \cup (\pi, \frac{4\pi}{3})$ and negative on $(\frac{2\pi}{3}, \pi) \cup (\frac{4\pi}{3}, 2\pi)$. So F rises on $(0, \frac{2\pi}{3})$, falls on $(\frac{2\pi}{3}, \pi)$ to its local minimum $F(\pi) > 0$, then mirrors that shape on $(\pi, 2\pi)$ back down to $F(2\pi) = 0$. Hence $F > 0$ throughout $(0, 2\pi)$.

So S holds iff k is an integer multiple of 2π.

Comparing with R:

$S \Rightarrow R$: every multiple of 2π is a multiple of π, so R is necessary for S.

R does not imply S: $k = \pi$ is a multiple of π but $F(\pi) > 0$, so R is not sufficient.

Therefore R is necessary but not sufficient for S.

Question 11

Tags: General Trigonometry, General Algebra · Difficulty: 7

Find the maximum value of

$$f(x) = \frac{2\sin^2 x + 10\cos x - 14}{\cos^2 x + 3\cos x - 10}$$

for $x \in \mathbb{R}$.

A $\frac{2}{3}$

B 1

C $\frac{5}{3}$

D 2

E $\frac{8}{3}$

F f has no maximum value

Solution 11

Answer: D

Let $u = \cos x$, so $u \in [-1, 1]$ and $\sin^2 x = 1 - u^2$.

The numerator becomes

$$2(1-u^2) + 10u - 14 = -2u^2 + 10u - 12 = -2(u-2)(u-3).$$

The denominator becomes

$$u^2 + 3u - 10 = (u-2)(u+5).$$

Since $u \in [-1, 1]$ we have $u \neq 2$, so the factor $(u-2)$ cancels:

$$f = \frac{-2(u-3)}{u+5} = \frac{6-2u}{u+5}.$$

Differentiating with respect to u:

$$\frac{df}{du} = \frac{-2(u+5) - (6-2u)}{(u+5)^2} = \frac{-16}{(u+5)^2} < 0,$$

so f is strictly decreasing in u on $[-1, 1]$. The maximum occurs at $u = -1$:

$$f_{\max} = \frac{6 - 2(-1)}{-1 + 5} = \frac{8}{4} = 2.$$

This is attained at $x = \pi$ (and any x with $\cos x = -1$).

Question 12

Tags: Logic Sufficiency, Differentiation · Difficulty: 7

Given curve $y = -\frac{1}{3}x^3 + a^2x + a$, has a local minimum in the second quadrant and intercepts x-axis at exactly one point, find the possible values of a.

A $0 < a < \sqrt{\dfrac{3}{2}}$

B Any a such that $a \neq 0$

C $-\sqrt{\dfrac{3}{2}} < a < \sqrt{\dfrac{3}{2}}$ with $a \neq 0$

D $-\sqrt{3} < a < 0$

E $0 < a < \sqrt{3}$

F $-\sqrt{3} < a < \sqrt{3}$ with $a \neq 0$

G There are no possible values of a.

Solution 12

Answer: A

Differentiate: $y' = -x^2 + a^2 = -(x-a)(x+a)$, so the stationary points are at $x = \pm a$ (with $a = 0$ there is no local minimum since $y' = -x^2$ doesn't change sign).

Since $y'' = -2x$, the local minimum is the stationary point where $y'' > 0$: $x = -a$ when $a > 0$, and $x = a$ when $a < 0$. Either way the local minimum sits at $x < 0$; for it to be in the second quadrant we additionally need $y > 0$ there.

Case $a > 0$. At $x = -a$,

$$y(-a) = \frac{a^3}{3} - a^3 + a = \frac{a(3-2a^2)}{3}.$$

With $a > 0$, this is positive iff $a^2 < \frac{3}{2}$, giving $0 < a < \sqrt{\frac{3}{2}}$.

Case $a < 0$. At $x = a$,

$$y(a) = -\frac{a^3}{3} + a^3 + a = \frac{a(3+2a^2)}{3}.$$

Here $3 + 2a^2 > 0$, so $y(a)$ has the same sign as a, hence is negative. No solutions.

Single x-intercept. The cubic has negative leading coefficient, so it runs from $+\infty$ down to $-\infty$; it crosses the x-axis exactly once iff the local minimum and local maximum lie on the same side of zero. For $0 < a < \sqrt{\frac{3}{2}}$ the local maximum value is

$$y(a) = \frac{a(3 + 2a^2)}{3} > 0,$$

matching the (positive) local minimum — both above the axis, so the single-intercept condition is automatic.

Hence the answer is **A**: $0 < a < \sqrt{\frac{3}{2}}$.

Question 13

Tags: Exponentials and Logarithms, Graphs of Functions · Difficulty: 7.5

Consider the curve

$$y = \log_{x-1} 4 \quad \text{for } x > 1,\ x \neq 2.$$

Which of the following statements about this curve are true?

I. $y > 0$ for every x in the domain.

II. y is a strictly decreasing function of x on the domain.

III. For every real number y, there exist a value x in the domain such that $y = \log_{x-1} 4$.

IV. $y = 0$ is an asymptote.

A I only

B II only

C I and II only

D I and III only

E II and IV only

F III and IV only

G I and IV only

H II and III only

Solution 13

Answer: E

Rewrite $y = \log_{x-1} 4$ as $(x-1)^y = 4$ and take natural log of both sides:

$$y \ln(x-1) = \ln 4.$$

Since $x \neq 2$ we have $\ln(x-1) \neq 0$, so

$$y = \frac{\ln 4}{\ln(x-1)}.$$

I. Take $x = \frac{3}{2}$. Then $y = \log_{1/2} 4 = -2$ since $\left(\frac{1}{2}\right)^{-2} = 4$. So $y < 0$ at this point, and **I is false**.

II. The function $\ln(x-1)$ is strictly increasing on $(1, \infty)$. On $(1, 2)$ it is negative and increases toward 0, so $\frac{1}{\ln(x-1)}$ strictly decreases (becoming more negative). On $(2, \infty)$ it is positive and increases, so $\frac{1}{\ln(x-1)}$ strictly decreases. Multiplying by the positive constant $\ln 4$ preserves this, so $y = \frac{\ln 4}{\ln(x-1)}$ strictly decreases on each part of the domain. Hence **II is true**.

III. For every x in the domain, $\ln(x-1) \neq 0$ and $\ln 4 \neq 0$, so $y = \frac{\ln 4}{\ln(x-1)} \neq 0$. Thus $y = 0$ is never attained, and **III is false**.

IV. As $x \to \infty$, $\ln(x-1) \to \infty$, so $y = \frac{\ln 4}{\ln(x-1)} \to 0$. Hence $y = 0$ is a horizontal asymptote, and **IV is true**.

The true statements are II and IV, giving option E.

Question 14

Tags: Logic Deduction, Logic Counterexample · Difficulty: 7.5

The function f is defined for positive integers and satisfies

$$f(1) = 1, \qquad f(2n) = f(n), \qquad f(2n+1) = f(n) + 1.$$

Consider the following three statements:

1 For every positive integer n, $f(2^n - 1) = n$.

2 For all positive integers m and n, $f(mn) = f(m) + f(n)$.

3 For every positive integer k, there exist infinitely many positive integers n such that $f(n) = k$.

Which of the above statements are true?

A none of them

B **1** only

C **2** only

D **3** only

E **1** and **2** only

F **1** and **3** only

G **2** and **3** only

H **1**, **2** and **3**

Solution 14

Answer: F

First observe that $f(n)$ counts the number of 1s in the binary representation of n. Indeed, doubling appends a 0 (so $f(2n) = f(n)$), and $2n+1$ has the same binary digits as n with an extra 1 appended (so $f(2n+1) = f(n) + 1$); this matches $f(1) = 1$.

Statement 1: $2^n - 1$ in binary is n consecutive 1s, so $f(2^n - 1) = n$. We can also verify directly: $f(2^n - 1) = f(2 \cdot 2^{n-1} - 1) = f(2(2^{n-1} - 1) + 1) = f(2^{n-1} - 1) + 1$, and $f(2^1 - 1) = f(1) = 1$, giving $f(2^n - 1) = n$ by induction. **True.**

Statement 2: Take $m = n = 3$. Then $f(3) = f(2 \cdot 1 + 1) = f(1) + 1 = 2$, but $f(9) = f(2 \cdot 4 + 1) = f(4) + 1 = f(2) + 1 = f(1) + 1 = 2$. So $f(mn) = f(9) = 2$ while $f(m) + f(n) = 4$. **False.**

Statement 3: For any $k \geqslant 1$, the numbers $2^N + (2^k - 1)$ for $N \geqslant k$ all have binary representation consisting of a leading 1, then $N - k$ zeros, then k ones, hence exactly k ones, so f-value k. Infinitely many such N exist. **True.**

So **1** and **3** are true; **2** is false. Answer: F.

Question 15

Tags: Logic Deduction · Difficulty: 8

In this question, n is a positive integer.

The following is an attempted proof of the false statement:

If n does not divide $k!$ for any positive integer k with $k < n$, then n is prime.

Which line contains the first error?

(1) Assume n does not divide $k!$ for any positive integer k with $k < n$.

(2) Suppose for contradiction that n is composite, so $n = ab$ for some integers a, b with $1 < a \leq b < n$.

(3) Case 1: $a < b$. Then a and b are distinct integers, each in the set $\{2, 3, \ldots, n-1\}$.

(4) Therefore a and b both appear as factors in the product $(n-1)! = 1 \cdot 2 \cdots (n-1)$, so $ab = n$ divides $(n-1)!$.

(5) Case 2: $a = b$, so $n = a^2$.

(6) Since $a \geq 2$, the integers a and $2a$ both satisfy $a \geq 2$ and $2a \geq 4$, so they are positive integers.

(7) Since $a \geq 2$, we have $a < a^2 = n$ and $2a < a^2 = n$, so a and $2a$ are distinct integers in the set $\{2, 3, \ldots, n-1\}$.

(8) Therefore a and $2a$ both appear as factors in $(n-1)!$, so $a \cdot 2a = 2a^2 = 2n$ divides $(n-1)!$, and in particular n divides $(n-1)!$.

(9) In both cases n divides $(n-1)!$, contradicting the assumption in line 1. Hence n is prime.

A Line 1

B Line 2

C Line 3

D Line 4

E Line 5

F Line 6

G Line 7

H Line 8

I Line 9

Solution 15

Answer: G

A counterexample to the statement is $n = 4$: the values $1! = 1$, $2! = 2$, $3! = 6$ are none divisible by 4, yet 4 is not prime. So the proof must contain an error.

Tracing $n = 4$ (with $a = b = 2$, falling into Case 2) through the proof: lines 1–6 hold for $n = 4$, $a = 2$.

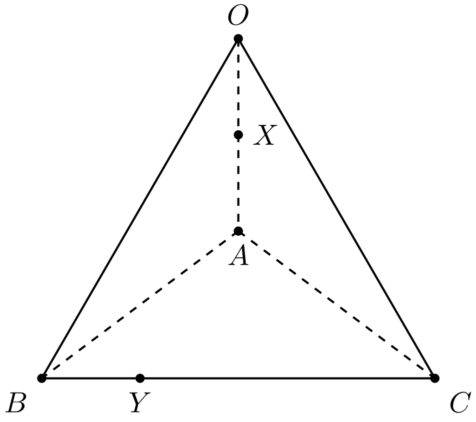
Line 7 asserts that $a \geq 2$ implies the strict inequality $2a < a^2 = n$. This is wrong: $2a < a^2$ rearranges to $a > 2$, i.e. $a \geq 3$, not $a \geq 2$. When $a = 2$ we have $2a = 4 = n$, so $2a$ is **not** in $\{2, 3, \ldots, n-1\} = \{2, 3\}$. The error is therefore the unjustified jump from $a \geq 2$ to the strict inequality $2a < n$ (the analogous claim $a < n$ is fine).

Once line 7 fails for $a = 2$, line 8 is unsupported (in $(n-1)! = 3! = 6$, the integer $2a = 4$ does not appear as a factor, and indeed $4 \nmid 6$), and the contradiction in line 9 evaporates. The error originates in line 7.

Question 16

Tags: Geometry · Difficulty: 8

A regular tetrahedron $OABC$ has every edge of length 12 metres. The point X is the midpoint of edge OA, and the point Y lies on edge BC with $BY = 3$ metres, as shown below.



What is the shortest distance, in metres, from X to Y measured entirely along the outer surface of the tetrahedron?

A $3\sqrt{3}$

B $3\sqrt{7}$

C $3\sqrt{13}$

D $3\sqrt{21}$

E 21

Solution 16

Answer: C

Each face is equilateral with side 12, so every face angle is 60°.

There are four candidate geodesics, each obtained by unfolding two adjacent faces flat about their shared edge: about OB, about OC, about AB, or about AC.

Unfolding about OB (faces OAB and OBC):

Place $O = (0,0)$ and $B = (12,0)$. In the unfolded plane, $A = (6, 6\sqrt{3})$ and $C = (6, -6\sqrt{3})$. Then

$$X = \text{midpoint}(OA) = (3, 3\sqrt{3}), \qquad Y = B + \frac{1}{4}(C - B) = (10.5,\ -1.5\sqrt{3}).$$

The straight-line distance is

$$XY^2 = (10.5 - 3)^2 + (-1.5\sqrt{3} - 3\sqrt{3})^2 = 7.5^2 + (4.5\sqrt{3})^2 = 56.25 + 60.75 = 117,$$

so $XY = \sqrt{117} = 3\sqrt{13}$. The straight line crosses the x-axis at $(8,0)$, which lies on edge OB, so the path is a valid surface geodesic.

Unfolding about AB (faces OAB and ABC): By the same calculation (the geometry is congruent), the distance is again $3\sqrt{13}$, with the path crossing edge AB.

Unfolding about OC (faces OAC and OBC): Place $O = (0,0)$, $C = (12,0)$, with $A = (6, 6\sqrt{3})$ and $B = (6, -6\sqrt{3})$. Now $X = (3, 3\sqrt{3})$ but $Y = B + \frac{1}{4}(C - B) = (7.5,\ -4.5\sqrt{3})$, giving

$$XY^2 = 4.5^2 + (7.5\sqrt{3})^2 = 20.25 + 168.75 = 189,$$

so $XY = 3\sqrt{21}$.

Unfolding about AC gives the same value $3\sqrt{21}$ by symmetry.

The shortest of $\{3\sqrt{13},\ 3\sqrt{21}\}$ is $3\sqrt{13}$.

Distractor analysis:

- $3\sqrt{3}$ comes from applying the cosine rule once on a single face with angle 60°: $\sqrt{36 + 9 - 36 \cdot \frac{1}{2}} = \sqrt{27}$. This ignores that X and Y are not on a common face.
- $3\sqrt{7}$ comes from the cosine rule with $\cos 120°$ but only $OX = 6$, $OY = 3$ on a single unfold step: $\sqrt{36 + 9 + 18} = \sqrt{63}$.
- $3\sqrt{21}$ is the length of the suboptimal unfolded route via edge OC or AC.
- 21 is the path along edges $X \to O \to B \to Y$ of total length $6 + 12 + 3$.

Question 17

Tags: Logic Deduction, Exponentials and Logarithms · Difficulty: 8.5

Let $a, b, c > 0$ with $a \neq 1$, $b \neq 1$ and $c \neq 1$. Consider the three equations

$$\log_a b = c, \qquad \log_b c = a, \qquad \log_c a = b.$$

Which one of the following statements about the solutions (a, b, c) of this system is correct?

A The equations specify a, b and c uniquely.

B The equations specify the product abc uniquely but have infinitely many solutions for (a, b, c).

C The equations specify exactly one of a, b, c uniquely but have infinitely many solutions for the other two.

D The equations have no solutions.

Solution 17

Answer: D

We are given $\log_a b = c$, $\log_b c = a$, and $\log_c a = b$. Rewrite each equation in exponential form:

$$b = a^c, \qquad c = b^a, \qquad a = c^b.$$

Since $a > 0$, $b > 0$, and $c > 0$, each logarithm is positive.

Therefore a and b are on the same side of 1, b and c are on the same side of 1, and c and a are on the same side of 1.

So either $a > 1$, $b > 1$, and $c > 1$, or $0 < a < 1$, $0 < b < 1$, and $0 < c < 1$.

Case 1: $a > 1$, $b > 1$, and $c > 1$. From $b = a^c$, since $a > 1$ and $c > 1$, we get $b > a$. From $c = b^a$, since $b > 1$ and $a > 1$, we get $c > b$. From $a = c^b$, since $c > 1$ and $b > 1$, we get $a > c$. Therefore $a > c > b > a$, which is impossible.

Case 2: $0 < a < 1$, $0 < b < 1$, and $0 < c < 1$. For any $0 < x < 1$ and $0 < r < 1$, we have $x^r > x$.

Therefore, from $b = a^c$, we get $b > a$. From $c = b^a$, we get $c > b$. From $a = c^b$, we get $a > c$.

Therefore $a > c > b > a$, which is again impossible. Hence the system has no solutions. The correct answer is **D**.

Question 18

Tags: Logic Deduction, Logic Negation · Difficulty: 8.5

A safe has three levers, A, B and C, each of which can be positioned either left or right at any particular time. The state of the safe (open or closed) depends only on the positions of these three levers. It is known that:

> **If** lever A is right and (lever B is left or lever C is right), **then** the safe is open.

Which one of the following statements **must** be true?

A If the safe is open, then lever A is right and either lever B is left or lever C is right.

B If the safe is closed, then lever A is left, and either lever B is right or lever C is left.

C If the safe is closed, then lever A is left, lever B is right and lever C is left.

D If the safe is closed, then either lever A is left, or both lever B is right and lever C is left.

E If lever A is left, or both lever B is right and lever C is left, then the safe is closed.

F If the safe is open, then either lever A is left, or both lever B is right and lever C is left.

Solution 18

Answer: D

Let P be the proposition $A = \text{right} \wedge (B = \text{left} \vee C = \text{right})$, and let Q be the proposition that the safe is open. The given information is $P \Rightarrow Q$.

The only deduction that **must** follow is the contrapositive: $\neg Q \Rightarrow \neg P$, i.e.

$$\text{safe closed} \;\Rightarrow\; \neg[A = \text{right} \wedge (B = \text{left} \vee C = \text{right})]\,.$$

Applying De Morgan's laws,

$$\neg[A = \text{right} \wedge (B = \text{left} \vee C = \text{right})] \;=\; (A = \text{left}) \vee \big[(B = \text{right}) \wedge (C = \text{left})\big].$$

So the safe being closed implies: lever A is left, or (both lever B is right and lever C is left). This is option D.

Why the others fail: **A** is the converse $Q \Rightarrow P$, which does not follow. **B** negates the AND/OR structure incorrectly: it keeps the outer connective as AND and the inner as OR, instead of swapping both via De Morgan. **C** over-negates by demanding all three lever positions simultaneously, ignoring that negating an AND only requires one conjunct to fail. **E** is the inverse $\neg P \Rightarrow \neg Q$, which does not follow. **F** is a scope error: it links the safe being open to the negation of P, whereas the contrapositive links the safe being **closed** to the negation of P.

Question 19

Tags: Integration, Sequences and Series · Difficulty: 9

Let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x. Compute

$$\int_0^{20} \lceil x \rceil \cdot 2^{\lceil x \rceil}\, dx.$$

A $20 \cdot 2^{21}$

B $19 \cdot 2^{21}$

C $18 \cdot 2^{20} + 2$

D $19 \cdot 2^{21} + 2$

E $19 \cdot 2^{20} + 2$

F $20 \cdot 2^{20} + 2$

Solution 19

Answer: D

On the interval $(k-1, k]$ we have $\lceil x \rceil = k$, so the integrand equals $k \cdot 2^k$ on each unit interval. Hence

$$\int_0^{20} \lceil x \rceil \cdot 2^{\lceil x \rceil}\, dx = \sum_{k=1}^{20} k \cdot 2^k.$$

To evaluate $S = \sum_{k=1}^{20} k \cdot 2^k$, use the standard trick of subtracting a shifted copy:

$$2S = \sum_{k=1}^{20} k \cdot 2^{k+1} = \sum_{j=2}^{21} (j-1) \cdot 2^j.$$

Then

$$S = 2S - S = -\sum_{j=2}^{20} 2^j + 20 \cdot 2^{21} - 1 \cdot 2^1.$$

The geometric sum is $\sum_{j=2}^{20} 2^j = 2^{21} - 4$, so

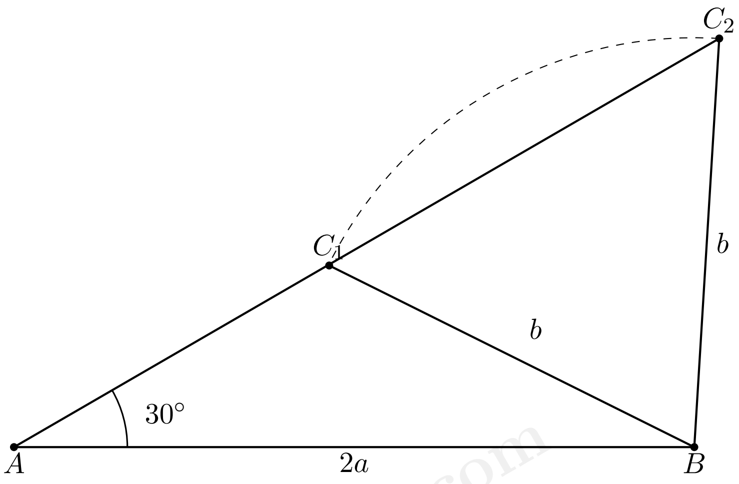
$$S = -(2^{21} - 4) + 20 \cdot 2^{21} - 2 = 19 \cdot 2^{21} + 2.$$

The answer is **D**.

Question 20

Tags: General Trigonometry, Geometry · Difficulty: 9

A triangle ABC is to be drawn with $AB = 2a$, $BC = b$, and $\angle BAC = 30^\circ$, where a and b are positive constants. For certain values of b, these conditions specify two distinct triangles ABC, with the third vertex at C_1 or C_2 as shown.



For which values of b is $\angle ABC$ acute in each of these two triangles?

A $a < b < 2a$

B $\dfrac{2\sqrt{3}}{3}a < b < 2a$

C $a < b < \dfrac{2\sqrt{3}}{3}a$

D $b < \dfrac{2\sqrt{3}}{3}a$

E $a < b < \sqrt{3}\,a$

F $\sqrt{3}\,a < b < 2a$

Solution 20

Answer: C

Label the sides so that $c = AB = 2a$ is opposite C and $b = BC$ is opposite $A = 30^\circ$. By the sine rule,

$$\sin C = \frac{c \sin A}{b} = \frac{2a \cdot \frac{1}{2}}{b} = \frac{a}{b}.$$

Step 1: Ambiguous (SSA) range. Two distinct triangles arise when there are two valid values of C in $(0^\circ, 180^\circ)$, namely C_1 acute and $C_2 = 180^\circ - C_1$ obtuse, both compatible with $A + C < 180^\circ$. We need: (i) $\sin C < 1$, i.e. $\frac{a}{b} < 1$, so $b > a$; (ii) the obtuse solution C_2 must satisfy $A + C_2 < 180^\circ$, i.e. $C_1 > 30^\circ$, equivalently $\sin C_1 > \sin 30^\circ = \frac{1}{2}$, so $\frac{a}{b} > \frac{1}{2}$, i.e. $b < 2a$.

Hence two triangles exist if and only if $a < b < 2a$.

Step 2: Both triangles have $\angle ABC$ acute. Let B_1, B_2 be the values of $\angle ABC$ in the two triangles. Then

$$\begin{aligned} B_1 &= 180^\circ - 30^\circ - C_1 = 150^\circ - C_1, \\ B_2 &= 180^\circ - 30^\circ - (180^\circ - C_1) = C_1 - 30^\circ. \end{aligned}$$

Since $30^\circ < C_1 < 90^\circ$ on the ambiguous range, $B_2 = C_1 - 30^\circ \in (0^\circ, 60^\circ)$ is automatically acute. The binding constraint is $B_1 < 90^\circ$:

$$150^\circ - C_1 < 90^\circ \iff C_1 > 60^\circ \iff \sin C_1 > \frac{\sqrt{3}}{2} \iff \frac{a}{b} > \frac{\sqrt{3}}{2} \iff b < \frac{2a}{\sqrt{3}} = \frac{2\sqrt{3}}{3}a.$$

Combine: $a < b < \frac{2\sqrt{3}}{3}a$.

Quick check at $b = 1.1a$: $\sin C_1 = 1/1.1 \approx 0.909$, $C_1 \approx 65.4^\circ$, so $B_1 \approx 84.6^\circ$ and $B_2 \approx 35.4^\circ$ — both acute. ✓ At $b = 1.2a$: $C_1 \approx 56.4^\circ$, $B_1 \approx 93.6^\circ$ — obtuse, fails. ✓