# JZ Mock Set A Paper 2

**Time:** 75 minutes

**Calculators:** not permitted

**Format:** 20 multiple-choice questions

**Average difficulty:** 6.95

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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## Question 1

Consider the equation $x^3 - 3px^2 + 4 = 0$, where $p$ is a real parameter.

Which of the following is a **sufficient** but **not necessary** condition on $p$ for this equation to have exactly one real root?

**A** $p > 1$

**B** $p \geq 1$

**C** $p \leq 1$

**D** $|p| \geq 1$

**E** $p^2 > 1$

**F** $-1 < p < 1$

**G** $p \in \mathbb{R}$

**H** $p < 1$

**Question 2**

What is the gradient of the curve

$$y = \frac{(\sqrt{x}+2)^3}{x\sqrt{x}}$$

at the point where $x = 4$?

**A** $-\frac{5}{2}$

**B** $-\frac{3}{2}$

**C** $-\frac{7}{2}$

**D** $\frac{7}{2}$

**E** $\frac{5}{2}$

**F** $\frac{3}{2}$

**Question 3**

A student attempts to prove the following statement.

**Claim:** Consider integers $a$ and $b$, where $a$ has remainder 1 when divided by 3, and $b$ has remainder 2 when divided by 3. Then $a^2 + b^2 + 1$ is always divisible by 6.

Consider the following attempt:

$$\text{Let } a = 3n + 1 \text{ and } b = 3n + 2 \qquad \text{(I)}$$

$$\text{then } a^2 + b^2 + 1 = (3n + 1)^2 + (3n + 2)^2 + 1 \qquad \text{(II)}$$

$$\text{so } a^2 + b^2 + 1 = 9n^2 + 6n + 1 + 9n^2 + 12n + 4 + 1 \qquad \text{(III)}$$

$$\text{so } a^2 + b^2 + 1 = 18n^2 + 18n + 6 = 6(3n^2 + 3n + 1) \qquad \text{(IV)}$$

$$\text{therefore } a^2 + b^2 + 1 \text{ is always divisible by 6.} \qquad \text{(V)}$$

Which of the following best describes this proof?

**A** The statement is true and the proof is completely correct.

**B** The statement is true but there is an error in the proof in line (I).

**C** The statement is true but there is an error in the proof in line (III).

**D** The statement is not true and the error first occurs in line (I).

**E** The statement is not true and the error first occurs in line (IV).

**F** The statement is not true and the error first occurs in line (V).

**Question 4**

Consider the following statement:

If a positive integer $N$ has the property that $N^2$ is divisible by 12, then $N$ is divisible by 12.

Which of the following are counterexamples to this statement?

I $N = 6$

II $N = 8$

III $N = 18$

IV $N = 24$

**A** none of them

**B** I only

**C** II only

**D** I and III only

**E** I, II and III only

**F** II and IV only

**G** I, III and IV only

**H** I, II, III and IV

**Question 5**

Sequence 1 is an arithmetic progression with first term 10 and common difference 4.

Sequence 2 is an arithmetic progression with first term 8 and common difference 6.

Sequence 3 is an arithmetic progression with first term 23 and common difference 5.

Some positive integers appear in all three of Sequence 1, Sequence 2 and Sequence 3. Let $N$ be the 17th such integer.

What is the remainder when $N$ is divided by 7?

**A** 0

**B** 1

**C** 2

**D** 3

**E** 4

**F** 5

**G** 6

**Question 6**

The real numbers $a$, $b$, $c$ and $d$ satisfy $a \leqslant b$, with $a$ and $b$ both non-zero, and $c < d$.

Which of the following statements are **necessarily** true?

**I** $\dfrac{1}{a^3} \geq \dfrac{1}{b^3}$

**II** $3^{-a} \geqslant 3^{-b}$

**III** $a(d-c) \leqslant b(d-c)$

**IV** $\dfrac{a}{c^2+1} \leqslant \dfrac{b}{d^2+1}$

**A** none of them

**B** **II** only

**C** **III** only

**D** **II** and **III** only

**E** **I**, **II** and **III** only

**F** **II**, **III** and **IV** only

**G** **I**, **II** and **IV** only

**H** **I**, **II**, **III** and **IV**

## Question 7

A student attempts to prove the following claim. **Claim:** For every positive integer $n$, if $n^2 + 2$ is prime, then $n$ is a multiple of 3. The student's argument is given line by line below.

I. Suppose $n$ is a positive integer such that $n^2 + 2$ is prime, but $n$ is not a multiple of 3.

II. Then the remainder of $n$ on division by 3 is 1 or 2.

III. In either case, $n^2$ leaves remainder 1 when divided by 3.

IV. Therefore $n^2 + 2$ leaves remainder 0 when divided by 3.

V. So 3 divides $n^2 + 2$.

VI. Hence $n^2 + 2 = 3k$ for some integer $k$.

VII. It follows that $n^2 + 2$ is composite and not prime.

VIII. This contradicts the assumption in (I) that $n^2 + 2$ is prime.

IX. Therefore $n$ is a multiple of 3.

Which one of the following statements about the argument is correct?

**A** The argument is completely correct.

**B** The first error in the argument is on line I.

**C** The first error in the argument is on line II.

**D** The first error in the argument is on line III.

**E** The first error in the argument is on line IV.

**F** The first error in the argument is on line V.

**G** The first error in the argument is on line VI.

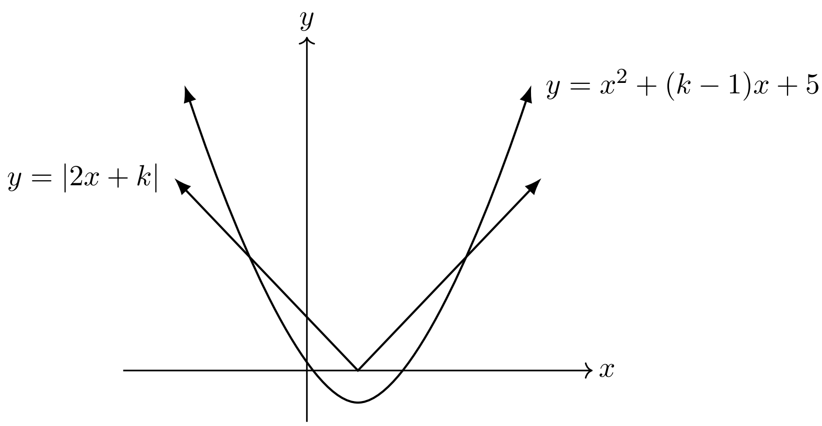
**H** The first error in the argument is on line VII.

**I** The first error in the argument is on line VIII.

**J** The first error in the argument is on line IX.

**Question 8**

For some values of $k$, the graphs of $y = |2x + k|$ and $y = x^2 + (k-1)x + 5$ intersect, as in the diagram below.




Find the complete set of values of $k$ for which the graph $y = |2x + k|$ lies strictly below the curve $y = x^2 + (k-1)x + 5$ for every real value of $x$.

**A** $1 - 2\sqrt{3} < k < 1 + 2\sqrt{3}$

**B** $k < 1 + 2\sqrt{7}$

**C** $1 - 2\sqrt{5} < k < 1 + 2\sqrt{5}$

**D** $1 - 2\sqrt{3} < k < 1 + 2\sqrt{5}$

**E** $k < 1 - 2\sqrt{3}$ or $k > 1 + 2\sqrt{3}$

**F** There are no such values of $k$.

**G** $k < 1 - 2\sqrt{5}$ or $k > 1 + 2\sqrt{5}$

**Question 9**

The region $R$ in the $(x, y)$-plane consists of all points satisfying **both**

$$|y - x^2| < 3 \qquad \text{and} \qquad x + y < 4.$$

Consider the following three claims about points in $R$.

**I**: For every $(x, y) \in R$, $y < 7$.

**II**: For every $(x, y) \in R$, $y > -3$.

**III**: For every $(x, y) \in R$, $x < 4$.

Which of the claims are true?

**A** None of them

**B** I only

**C** II only

**D** III only

**E** I and II only

**F** I and III only

**G** II and III only

**H** I, II and III

**Question 10**

Let $k$ be a real number, and consider the following two statements.

$R$: "$k$ is an integer multiple of $\pi$".

$S$: "$\int_0^k \left(\sin x + \sin^3(2x)\right)\, dx = 0$".

Which of the following best describes the logical relationship between $R$ and $S$?

**A** $R$ is necessary and sufficient for $S$.

**B** $R$ is necessary but not sufficient for $S$.

**C** $R$ is sufficient but not necessary for $S$.

**D** $R$ is neither necessary nor sufficient for $S$.

## Question 11

Find the maximum value of

$$f(x) = \frac{2\sin^2 x + 10\cos x - 14}{\cos^2 x + 3\cos x - 10}$$

for $x \in \mathbb{R}$.

**A** $\dfrac{2}{3}$

**B** $1$

**C** $\dfrac{5}{3}$

**D** $2$

**E** $\dfrac{8}{3}$

**F** $f$ has no maximum value

**Question 12**

Given curve $y = -\frac{1}{3}x^3 + a^2x + a$, has a local minimum in the second quadrant and intercepts $x$-axis at exactly one point, find the possible values of $a$.

**A** $0 < a < \sqrt{\frac{3}{2}}$

**B** Any $a$ such that $a \neq 0$

**C** $-\sqrt{\frac{3}{2}} < a < \sqrt{\frac{3}{2}}$ with $a \neq 0$

**D** $-\sqrt{3} < a < 0$

**E** $0 < a < \sqrt{3}$

**F** $-\sqrt{3} < a < \sqrt{3}$ with $a \neq 0$

**G** There are no possible values of $a$.

**Question 13**

Consider the curve

$$y = \log_{x-1} 4 \quad \text{for } x > 1,\ x \neq 2.$$

Which of the following statements about this curve are true?

**I.** $y > 0$ for every $x$ in the domain.

**II.** $y$ is a strictly decreasing function of $x$ on the domain.

**III.** For every real number $y$, there exist a value $x$ in the domain such that $y = \log_{x-1} 4$.

**IV.** $y = 0$ is an asymptote.

**A** I only

**B** II only

**C** I and II only

**D** I and III only

**E** II and IV only

**F** III and IV only

**G** I and IV only

**H** II and III only

## Question 14

The function $f$ is defined for positive integers and satisfies

$$f(1) = 1, \qquad f(2n) = f(n), \qquad f(2n+1) = f(n) + 1.$$

Consider the following three statements:

**1** For every positive integer $n$, $f(2^n - 1) = n$.

**2** For all positive integers $m$ and $n$, $f(mn) = f(m) + f(n)$.

**3** For every positive integer $k$, there exist infinitely many positive integers $n$ such that $f(n) = k$.

Which of the above statements are true?

**A** none of them

**B** **1** only

**C** **2** only

**D** **3** only

**E** **1** and **2** only

**F** **1** and **3** only

**G** **2** and **3** only

**H** **1**, **2** and **3**

**Question 15**

In this question, $n$ is a positive integer.

The following is an attempted proof of the false statement:

If $n$ does not divide $k!$ for any positive integer $k$ with $k < n$, then $n$ is prime.

Which line contains the first error?

(1) Assume $n$ does not divide $k!$ for any positive integer $k$ with $k < n$.

(2) Suppose for contradiction that $n$ is composite, so $n = ab$ for some integers $a$, $b$ with $1 < a \leq b < n$.

(3) Case 1: $a < b$. Then $a$ and $b$ are distinct integers, each in the set $\{2, 3, \ldots, n-1\}$.

(4) Therefore $a$ and $b$ both appear as factors in the product $(n-1)! = 1 \cdot 2 \cdots (n-1)$, so $ab = n$ divides $(n-1)!$.

(5) Case 2: $a = b$, so $n = a^2$.

(6) Since $a \geq 2$, the integers $a$ and $2a$ both satisfy $a \geq 2$ and $2a \geq 4$, so they are positive integers.

(7) Since $a \geq 2$, we have $a < a^2 = n$ and $2a < a^2 = n$, so $a$ and $2a$ are distinct integers in the set $\{2, 3, \ldots, n-1\}$.

(8) Therefore $a$ and $2a$ both appear as factors in $(n-1)!$, so $a \cdot 2a = 2a^2 = 2n$ divides $(n-1)!$, and in particular $n$ divides $(n-1)!$.

(9) In both cases $n$ divides $(n-1)!$, contradicting the assumption in line 1. Hence $n$ is prime.

**A** Line 1

**B** Line 2

**C** Line 3

**D** Line 4

**E** Line 5

**F** Line 6

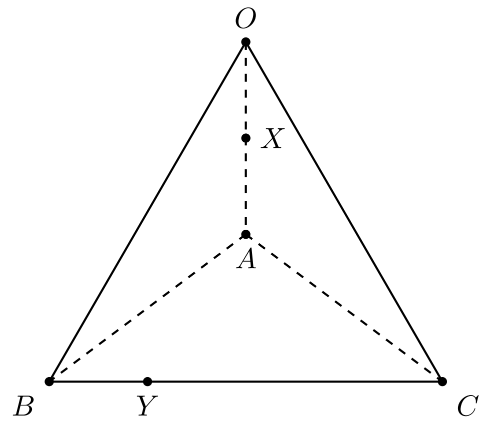
**G** Line 7

**H** Line 8

**I** Line 9

**Question 16**

A regular tetrahedron $OABC$ has every edge of length 12 metres. The point $X$ is the midpoint of edge $OA$, and the point $Y$ lies on edge $BC$ with $BY = 3$ metres, as shown below.




What is the shortest distance, in metres, from $X$ to $Y$ measured entirely along the outer surface of the tetrahedron?

**A** $3\sqrt{3}$

**B** $3\sqrt{7}$

**C** $3\sqrt{13}$

**D** $3\sqrt{21}$

**E** 21

**Question 17**

Let $a, b, c > 0$ with $a \neq 1$, $b \neq 1$ and $c \neq 1$. Consider the three equations

$$\log_a b = c, \qquad \log_b c = a, \qquad \log_c a = b.$$

Which one of the following statements about the solutions $(a, b, c)$ of this system is correct?

**A** The equations specify $a$, $b$ and $c$ uniquely.

**B** The equations specify the product $abc$ uniquely but have infinitely many solutions for $(a, b, c)$.

**C** The equations specify exactly one of $a$, $b$, $c$ uniquely but have infinitely many solutions for the other two.

**D** The equations have no solutions.

## Question 18

A safe has three levers, $A$, $B$ and $C$, each of which can be positioned either left or right at any particular time. The state of the safe (open or closed) depends only on the positions of these three levers. It is known that:

**If** lever $A$ is right and (lever $B$ is left or lever $C$ is right), **then** the safe is open.

Which one of the following statements **must** be true?

**A** If the safe is open, then lever $A$ is right and either lever $B$ is left or lever $C$ is right.

**B** If the safe is closed, then lever $A$ is left, and either lever $B$ is right or lever $C$ is left.

**C** If the safe is closed, then lever $A$ is left, lever $B$ is right and lever $C$ is left.

**D** If the safe is closed, then either lever $A$ is left, or both lever $B$ is right and lever $C$ is left.

**E** If lever $A$ is left, or both lever $B$ is right and lever $C$ is left, then the safe is closed.

**F** If the safe is open, then either lever $A$ is left, or both lever $B$ is right and lever $C$ is left.

**Question 19**

Let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to $x$. Compute

$$\int_0^{20} \lceil x \rceil \cdot 2^{\lceil x \rceil} \, dx.$$

**A** $20 \cdot 2^{21}$

**B** $19 \cdot 2^{21}$

**C** $18 \cdot 2^{20} + 2$

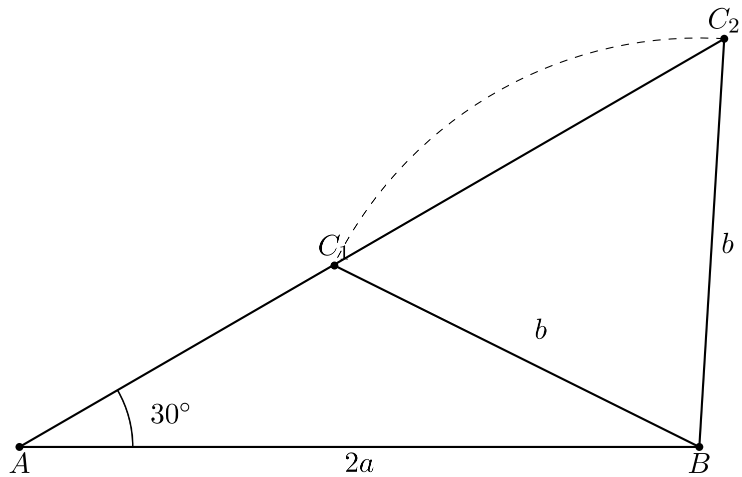
**D** $19 \cdot 2^{21} + 2$

**E** $19 \cdot 2^{20} + 2$

**F** $20 \cdot 2^{20} + 2$

**Question 20**

A triangle $ABC$ is to be drawn with $AB = 2a$, $BC = b$, and $\angle BAC = 30°$, where $a$ and $b$ are positive constants. For certain values of $b$, these conditions specify two distinct triangles $ABC$, with the third vertex at $C_1$ or $C_2$ as shown.






For which values of $b$ is $\angle ABC$ acute in each of these two triangles?

**A** $a < b < 2a$

**B** $\frac{2\sqrt{3}}{3}a < b < 2a$

**C** $a < b < \frac{2\sqrt{3}}{3}a$

**D** $b < \frac{2\sqrt{3}}{3}a$

**E** $a < b < \sqrt{3}\,a$

**F** $\sqrt{3}\,a < b < 2a$