JZ Mock Set A Paper 1

Solutions

Time: 75 minutes

Calculators: not permitted

Format: 20 multiple-choice questions

Average difficulty: 6.975

This is a TMUA-style mock paper modelled on the Test of Mathematics for University Admission. The TMUA is used in admissions for mathematics, economics, computer science, and engineering courses at universities including Cambridge, Oxford, Imperial College London, UCL, LSE, Warwick, and Durham.

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Question 1

Tags: Area Integration · Difficulty: 5.5

Find the finite area enclosed between the curve $y = |x^2 - 9|$ and the line $y = 16$.

A $\dfrac{104}{3}$

B $\dfrac{142}{3}$

C $\dfrac{284}{3}$

D $\dfrac{460}{3}$

E $\dfrac{500}{3}$

F 60

Solution 1

Answer: C

The curve $y = |x^2 - 9|$ is even, so the enclosed region is symmetric in the y-axis; compute the area for $x \geq 0$ and double. The modulus changes formula at $x = 3$: for $0 \leq x \leq 3$, $y = 9 - x^2$; for $x \geq 3$, $y = x^2 - 9$. The peak of the inner piece is 9, which is below 16, so the line $y = 16$ lies strictly above the curve on the inner interval. The line meets the outer piece where $x^2 - 9 = 16$, i.e. $x = 5$ (and $x = -5$ by symmetry).

Area

$$
\begin{aligned}
&= 2\left[\int_0^3 \big(16 - (9 - x^2)\big)\,dx + \int_3^5 \big(16 - (x^2 - 9)\big)\,dx\right] \\
&= 2\left[\int_0^3 (7 + x^2)\,dx + \int_3^5 (25 - x^2)\,dx\right] \\
&= 2\left[\big(21 + 9\big) + \big(125 - \frac{125}{3} - 75 + 9\big)\right] \\
&= 2\left[30 + \frac{52}{3}\right] \\
&= 2 \cdot \frac{142}{3} = \frac{284}{3}.
\end{aligned}
$$

Question 2

Tags: Exponentials and Logarithms · Difficulty: 6

Find the sum of all real values of x for which there exist positive real numbers y and z satisfying the simultaneous equations

$$\log_2(x^2yz) = 3, \qquad \log_2(xz) = 1, \qquad (\log_2 y)(\log_2 z) = 2.$$

A 3

B $\frac{9}{4}$

C $\frac{9}{2}$

D 7

E 8

F 9

G $\frac{63}{4}$

H 17

Solution 2

Answer: F

Let $X = \log_2 x$, $Y = \log_2 y$, $Z = \log_2 z$. The three equations become

$$2X + Y + Z = 3, \qquad X + Z = 1, \qquad YZ = 2.$$

From the second equation, $X = 1 - Z$. Substituting into the first gives

$$2(1 - Z) + Y + Z = 3 \implies Y = 1 + Z.$$

The third equation now reads $(1 + Z)Z = 2$, i.e. $Z^2 + Z - 2 = 0$, so $(Z + 2)(Z - 1) = 0$ and $Z \in \{-2, 1\}$.

Case $Z = 1$: $X = 0$, $Y = 2$, giving $(x, y, z) = (1, 4, 2)$.

Case $Z = -2$: $X = 3$, $Y = -1$, giving $(x, y, z) = (8, \frac{1}{2}, \frac{1}{4})$.

In each case the values of y and z are positive, so both solutions are valid. The real values of x are 1 and 8, with sum $1 + 8 = 9$.

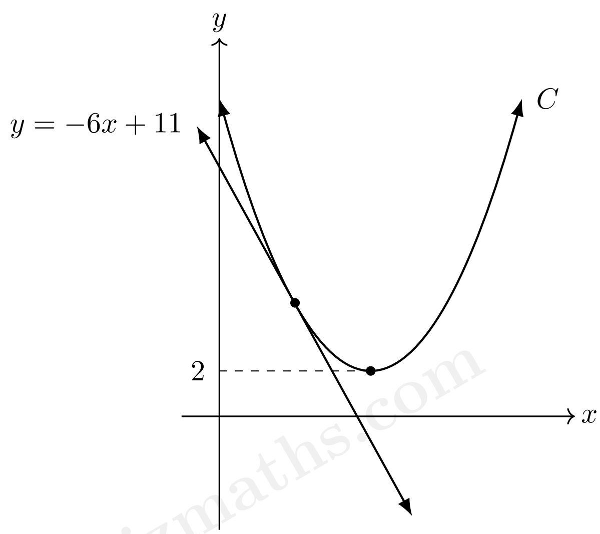
Question 3

Tags: General Algebra · Difficulty: 6

The curve C has equation $y = px^2 + qx + r$ where p, q and r are constants with $p > 0$.

C is tangent to the line $y = -6x + 11$ at the point where $x = 1$, and the minimum value of y on C is 2.

Find the value of p.



A 1

B 2

C 3

D 4

E 5

Solution 3

Answer: C

Tangency at $x = 1$ gives two conditions: (i) C passes through $(1, -6 + 11) = (1, 5)$, so $p + q + r = 5$, and (ii) the gradient of C at $x = 1$ equals -6, i.e. $2p + q = -6$.

From (ii): $q = -6 - 2p$. Substituting into (i): $r = 5 - p - q = 5 - p - (-6 - 2p) = 11 + p$.

The minimum value of $px^2 + qx + r$ for $p > 0$ is $r - \frac{q^2}{4p}$, so

$$(11 + p) - \frac{(6 + 2p)^2}{4p} = 2.$$

Multiplying by $4p$ and simplifying: $4p(11 + p) - (6 + 2p)^2 = 8p$, i.e. $44p + 4p^2 - 36 - 24p - 4p^2 = 8p$, giving $20p - 36 = 8p$, so $12p = 36$ and $p = 3$.

The answer is **C**.

Question 4

Tags: Exponentials and Logarithms · Difficulty: 6

Let x, y and z be positive real numbers satisfying

$$18^{x+z}6^{x+y+5} = 12^{y-2}18^5 6^{z+2}.$$

Find the value of $x + y + z$.

A 5

B 6

C 7

D 8

E 9

F 10

Solution 4

Answer: C

Write the bases as products of primes: $6 = 2 \cdot 3$, $12 = 2^2 \cdot 3$, $18 = 2 \cdot 3^2$. Reduce each side to the form $2^A \cdot 3^B$:

$$\text{LHS} = 2^{(x+z)+(x+y+5)} \cdot 3^{2(x+z)+(x+y+5)} = 2^{2x+y+z+5} \cdot 3^{3x+y+2z+5},$$

$$\text{RHS} = 2^{2(y-2)+5+(z+2)} \cdot 3^{(y-2)+10+(z+2)} = 2^{2y+z+3} \cdot 3^{y+z+10}.$$

Equating exponents prime-by-prime:

$$2x + y + z + 5 = 2y + z + 3 \implies y = 2x + 2,$$

$$3x + y + 2z + 5 = y + z + 10 \implies z = 5 - 3x.$$

Adding gives $x + y + z = x + (2x + 2) + (5 - 3x) = 7$. The two equations do not pin down x, y, z individually (any $x \in (0, 5/3)$ with $y = 2x + 2$ and $z = 5 - 3x$ gives positive reals satisfying the original equation), but the sum is forced. The answer is C.

Question 5

Tags: Graphs of Functions, General Functions · Difficulty: 6

A continuous function f defined on the real numbers has range $[-3, 2]$. (That is, every real value in the closed interval $[-3, 2]$ is attained by f at some real x, and no values outside this interval are attained.)

What is the difference between the maximum and the minimum of

$$(f(x))^2 + 3\,f(x) - 1$$

as x varies over the real numbers?

A $\frac{9}{4}$

B $\frac{13}{4}$

C 9

D $\frac{25}{4}$

E 10

F $\frac{41}{4}$

G $\frac{45}{4}$

H $\frac{49}{4}$

Solution 5

Answer: H

Let $t = f(x)$. Because f is continuous and its range is the full closed interval $[-3, 2]$, t takes every value in $[-3, 2]$ and no others. The problem reduces to finding the maximum and minimum of

$$g(t) = t^2 + 3t - 1$$

on the closed interval $t \in [-3, 2]$.

Complete the square:

$$g(t) = \left(t + \frac{3}{2}\right)^2 - \frac{9}{4} - 1 = \left(t + \frac{3}{2}\right)^2 - \frac{13}{4}.$$

The vertex is at $t = -\frac{3}{2}$, which lies inside $[-3, 2]$, so the minimum on the interval is

$$g\left(-\frac{3}{2}\right) = -\frac{13}{4}.$$

The maximum on a closed interval where the vertex is interior must occur at an endpoint. Evaluate both:

$$g(-3) = 9 - 9 - 1 = -1, \qquad g(2) = 4 + 6 - 1 = 9.$$

So the maximum is $g(2) = 9$.

The required difference is

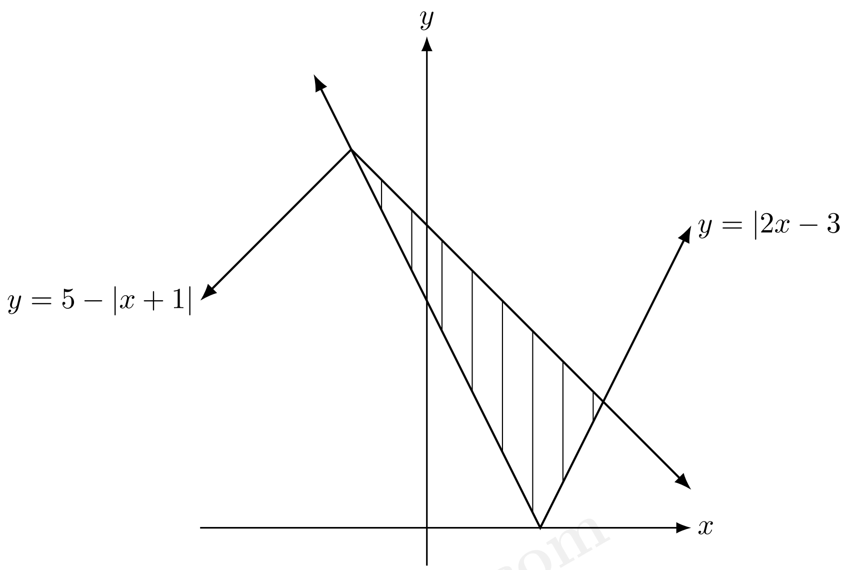
$$9 - \left(-\frac{13}{4}\right) = \frac{36}{4} + \frac{13}{4} = \frac{49}{4}.$$

The answer is **H**.

Question 6

Tags: Integration, Geometry · Difficulty: 6

Find the area of the region enclosed between the curves $y = |2x-3|$ and $y = 5 - |x+1|$.



A $\dfrac{25}{8}$

B $\dfrac{50}{9}$

C $\dfrac{25}{6}$

D 5

E $\dfrac{25}{3}$

Solution 6

Answer: C

The two graphs are a V-shape with vertex at $\left(\frac{3}{2}, 0\right)$ and slopes ± 2, and an inverted V with vertex at $(-1, 5)$ and slopes ± 1. There are two relevant breakpoints in x: $x = -1$ and $x = \frac{3}{2}$.

Intersections. Solve $|2x-3| = 5 - |x+1|$ by case.

For $x \geq \frac{3}{2}$ (so also $x \geq -1$):

$$2x - 3 = 5 - (x + 1) = 4 - x \implies 3x = 7 \implies x = \frac{7}{3},\ y = \frac{5}{3}.$$

For $-1 \leq x \leq \frac{3}{2}$:

$$3 - 2x = 5 - (x + 1) = 4 - x \implies x = -1,\ y = 5.$$

For $x < -1$:

$$3 - 2x = 5 - [-(x + 1)] = 6 + x \implies x = -1,$$

which is the boundary case already captured. A check at $x = -2$ shows $|2x - 3| = 7 > 4 = 5 - |x + 1|$, so the V lies above the inverted V to the left of $x = -1$ and no further enclosed region exists.

Thus the enclosed region runs from $x = -1$ to $x = \frac{7}{3}$, with the inverted V on top.

Area. Split at $x = \frac{3}{2}$ where the lower curve changes formula. On $\left[-1, \frac{7}{3}\right]$ the upper curve simplifies to $5 - (x + 1) = 4 - x$.

On $\left[-1, \frac{3}{2}\right]$ the integrand is $(4 - x) - (3 - 2x) = 1 + x$:

$$\int_{-1}^{3/2} (1 + x)\, dx = \left[x + \frac{x^2}{2}\right]_{-1}^{3/2} = \frac{21}{8} - \left(-\frac{1}{2}\right) = \frac{25}{8}.$$

On $\left[\frac{3}{2}, \frac{7}{3}\right]$ the integrand is $(4 - x) - (2x - 3) = 7 - 3x$:

$$\int_{3/2}^{7/3} (7 - 3x)\, dx = \left[7x - \frac{3x^2}{2}\right]_{3/2}^{7/3} = \frac{49}{6} - \frac{57}{8} = \frac{25}{24}.$$

Total area $= \frac{25}{8} + \frac{25}{24} = \frac{75}{24} + \frac{25}{24} = \frac{100}{24} = \frac{25}{6}$.

Cross-check. The region is a triangle with vertices $(-1, 5)$, $\left(\frac{3}{2}, 0\right)$ and $\left(\frac{7}{3}, \frac{5}{3}\right)$. By the shoelace formula,

$$\text{Area} = \frac{1}{2}\left|(-1)\left(0 - \frac{5}{3}\right) + \frac{3}{2}\left(\frac{5}{3} - 5\right) + \frac{7}{3}(5 - 0)\right| = \frac{1}{2}\left|\frac{5}{3} - 5 + \frac{35}{3}\right| = \frac{1}{2} \cdot \frac{25}{3} = \frac{25}{6}.$$

Question 7

Tags: Differentiation · Difficulty: 6.5

The function

$$f(x) = \frac{1}{4}x^{4/3} + \sqrt[3]{x} + \frac{3}{\sqrt[3]{x^2}}$$

is defined for all $x \neq 0$. The complete set of values of x for which f is increasing is given by

A $-3 \leq x < 0,\ x \geq 2$

B $x \leq -3,\ 0 < x \leq 2$

C $x \geq 2$

D $-3 < x < 0,\ x > 2$

E $-3 \leq x \leq 2,\ x \neq 0$

F $-3 \leq x \leq 0,\ x \geq 2$

Solution 7

Answer: A

Write $f(x) = \frac{1}{4}x^{4/3} + x^{1/3} + 3x^{-2/3}$. Differentiating term by term:

$$f'(x) = \frac{1}{3}x^{1/3} + \frac{1}{3}x^{-2/3} - 2x^{-5/3}.$$

Factor out $\frac{1}{3}x^{-5/3}$ (the lowest power):

$$f'(x) = \frac{1}{3}x^{-5/3}\big(x^2 + x - 6\big) = \frac{1}{3}x^{-5/3}(x+3)(x-2).$$

The function is increasing where $f'(x) \geq 0$. Since $x^{-5/3}$ has the same sign as x, analyse the sign of $x^{-5/3}(x+3)(x-2)$ on each interval (recall $x = 0$ is excluded from the domain). For $x < -3$ the signs give $(-)(-)(-) = -$, so f is decreasing. For $-3 < x < 0$ the signs give $(-)(+)(-) = +$, so f is increasing. For $0 < x < 2$ the signs give $(+)(+)(-) = -$, so f is decreasing. For $x > 2$ the signs give $(+)(+)(+) = +$, so f is increasing. At $x = -3$ and $x = 2$, $f'(x) = 0$, and these are interior points of intervals on which f is increasing on each side, so they are included. The point $x = 0$ is not in the domain and must be excluded.

Hence the complete increasing set is $-3 \leq x < 0$ together with $x \geq 2$.

Question 8

Tags: Polynomial Expansions · Difficulty: 6.5

In the expansion of $(a + bx)^7$, the coefficient of x^5 together with 2 times the coefficient of x^4 equals 5 times the coefficient of x^3. Given that a and b are positive integers, find the smallest possible value of $a + b$.

A 3

B 5

C 6

D 10

E 8

F 16

G 9

Solution 8

Answer: E

The relevant binomial coefficients in $(a + bx)^7$ are

$$\binom{7}{3}a^4b^3 = 35a^4b^3, \quad \binom{7}{4}a^3b^4 = 35a^3b^4, \quad \binom{7}{5}a^2b^5 = 21a^2b^5$$

for x^3, x^4, x^5 respectively. The given condition is

$$21a^2b^5 + 2(35a^3b^4) = 5(35a^4b^3),$$

i.e. $21a^2b^5 + 70a^3b^4 = 175a^4b^3$. Since $a, b > 0$, dividing by $7a^2b^3$ gives

$$3b^2 + 10ab - 25a^2 = 0,$$

which factorises as $(3b - 5a)(b + 5a) = 0$. Since $a, b > 0$ the second factor is positive, so $3b = 5a$. As $\gcd(3, 5) = 1$, the smallest positive integer solution is $a = 3$, $b = 5$, giving $a + b = 8$. The answer is E.

Question 9

Tags: General Number of Solutions, General Algebra · Difficulty: 6.5

For how many values of a is the equation

$$(x-a)(x^2+ax+a)=0$$

satisfied by exactly two distinct values of x?

A 0

B 1

C 2

D 3

E 4

F more than 4

Solution 9

Answer: C

The roots are $x=a$ and the roots of $Q(x)=x^2+ax+a=0$.

The discriminant of Q is $\Delta=a^2-4a=a(a-4)$, so Q has a repeated root iff $a=0$ or $a=4$.

Coincidence: $x=a$ is also a root of Q iff $Q(a)=a^2+a^2+a=2a^2+a=a(2a+1)=0$, i.e. $a=0$ or $a=-1/2$.

For exactly 2 distinct values of x, exactly one coincidence/multiplicity must occur (collapsing 3 generic roots to 2):

(i) $a=4$: $Q(x)=x^2+4x+4=(x+2)^2$, repeated root -2; combined with $x=a=4$. Distinct roots: $\{-2,4\}$. Two distinct. ✓

(ii) $a=-1/2$: $Q(x)=x^2-x/2-1/2$ has roots 1 and $-1/2$ (since $\Delta=1/4+2=9/4$, giving $x=(1/2\pm 3/2)/2$); $x=a=-1/2$ coincides with one quadratic root. Distinct roots: $\{1,-1/2\}$. Two distinct. ✓

(iii) $a = 0$: equation becomes $x \cdot x^2 = x^3 = 0$, giving only $x = 0$ — here both coincidence and repeated-root conditions occur simultaneously, collapsing all the way to one distinct value, not two. $\times$

For any other a: Q has two distinct roots and $x = a$ is distinct from both, giving three distinct values of x. $\times$

Total: 2 values of a (namely $a = 4$ and $a = -1/2$).

Question 10

Tags: Sequences and Series · Difficulty: 7

A convergent geometric progression has first term a and common ratio r. The sum to infinity of the progression is 8, and the sum to infinity of the series obtained by alternating the signs of the terms,

$$a - ar + ar^2 - ar^3 + \cdots,$$

is 24. What is the sum to infinity of the series whose terms are the cubes of the terms of the original progression?

A 192

B 512

C 1536

D 1728

E $\dfrac{13824}{7}$

F The series of cubes does not converge.

Solution 10

Answer: C

Let the original progression have first term a and common ratio r with $|r| < 1$. Then

$$\frac{a}{1-r} = 8 \qquad \text{and} \qquad \frac{a}{1+r} = 24.$$

Dividing the first by the second gives $\frac{1+r}{1-r} = \frac{8}{24} = \frac{1}{3}$, so $3(1+r) = 1 - r$, hence $4r = -2$ and $r = -\frac{1}{2}$.

Substituting back: $a = 8(1-r) = 8 \cdot \frac{3}{2} = 12$.

The series of cubes has first term $a^3 = 1728$ and common ratio $r^3 = -\frac{1}{8}$, with $|r^3| < 1$ so it converges. Its sum is

$$\frac{a^3}{1-r^3} = \frac{1728}{1 - \left(-\frac{1}{8}\right)} = \frac{1728}{\frac{9}{8}} = 1728 \cdot \frac{8}{9} = 1536.$$

Question 11

Tags: Differentiation, General Number of Solutions · Difficulty: 7

Find the set of values of k for which the equation

$$3x^4 - 8x^3 - 6x^2 + 24x + k = 0$$

has four distinct real solutions.

A $-13 < k < -8$

B $-13 < k < 19$

C $-19 < k < 13$

D $-8 < k < 19$

E $8 < k < 13$

F $k < -8$

G $k > -13$

H There are no such values of k.

Solution 11

Answer: A

Let $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + k$. Then

$$f'(x) = 12x^3 - 24x^2 - 12x + 24 = 12(x-2)(x-1)(x+1),$$

so the stationary points are at $x = -1$, $x = 1$ and $x = 2$.

Evaluating f at each:

$$\begin{aligned} f(-1) &= 3 + 8 - 6 - 24 + k = -19 + k, \\ f(1) &= 3 - 8 - 6 + 24 + k = 13 + k, \\ f(2) &= 48 - 64 - 24 + 48 + k = 8 + k. \end{aligned}$$

Since the leading coefficient is positive, f decreases on $(-\infty, -1)$, increases on $(-1, 1)$, decreases on $(1, 2)$ and increases on $(2, \infty)$, so $x = -1$ and $x = 2$ are local minima and $x = 1$ is a local maximum.

For the quartic to cross the x-axis four times we need both local minima strictly below the axis and the local maximum strictly above:

$$f(-1) < 0, \qquad f(1) > 0, \qquad f(2) < 0,$$

i.e.

$$k < 19, \qquad k > -13, \qquad k < -8.$$

The binding constraints are $k > -13$ and $k < -8$, giving $\boxed{-13 < k < -8}$, which is option **A**.

Question 12

Tags: Integration · Difficulty: 7

A function f satisfies

$$\int_0^2 f(x+1)\,dx = 1, \qquad \int_0^2 f(2-x)\,dx = 2, \qquad \int_0^{3/2} f(2x)\,dx = 3.$$

Find the value of $\int_1^2 f(x)\,dx$.

A -6

B -3

C 0

D 1

E 3

F 4

G 5

H 6

Solution 12

Answer: B

Let $A = \int_0^1 f$, $B = \int_1^2 f$, $C = \int_2^3 f$.

Equation 1. Substitute $u = x+1$ in $\int_0^2 f(x+1)\,dx$. Then $du = dx$ and the limits map $0 \to 1$, $2 \to 3$, giving

$$\int_1^3 f(u)\,du = B + C = 1.$$

Equation 2. Substitute $u = 2-x$ in $\int_0^2 f(2-x)\,dx$. Then $du = -dx$ and the limits map $0 \to 2$, $2 \to 0$, giving

$$\int_2^0 f(u)(-du) = \int_0^2 f(u)\,du = A + B = 2.$$

Equation 3. Substitute $u = 2x$ in $\int_0^{3/2} f(2x)\,dx$. Then $du = 2\,dx$ and the limits map $0 \to 0$, $3/2 \to 3$, giving

$$\frac{1}{2}\int_0^3 f(u)\,du = 3, \quad \text{so} \quad A + B + C = 6.$$

Solve. Subtracting Equation 1 from Equation 3 gives $A = 6 - 1 = 5$. Equation 2 then gives $B = 2 - A = -3$. (Check: $C = 1 - B = 4$, and $A + B + C = 5 - 3 + 4 = 6$.) Hence $\int_1^2 f(x)\,dx = -3$.

Question 13

Tags: General Algebra · Difficulty: 7

The equation $5x^2 + bx + 7 = 0$ has two real roots in the ratio $2 : 5$. Find the value of b^2.

A $\dfrac{343}{2}$

B $\dfrac{343}{10}$

C $\dfrac{343}{50}$

D $\dfrac{1715}{2}$

E 1225

Solution 13

Answer: A

Let the roots be $\alpha = 2k$ and $\beta = 5k$ for some non-zero k, so that $\alpha : \beta = 2 : 5$.

By Vieta's formulas applied to $5x^2 + bx + 7 = 0$:

$$\alpha\beta = \frac{7}{5}, \qquad \alpha + \beta = -\frac{b}{5}.$$

From the product: $10k^2 = \frac{7}{5}$, hence $k^2 = \frac{7}{50}$.

From the sum: $7k = -\frac{b}{5}$, hence $b = -35k$ and so $b^2 = 1225\,k^2$.

Therefore

$$b^2 = 1225 \cdot \frac{7}{50} = \frac{8575}{50} = \frac{343}{2}.$$

A quick check on reality of the roots: the discriminant is $b^2 - 4 \cdot 5 \cdot 7 = \frac{343}{2} - 140 = \frac{63}{2} > 0$, so the roots are indeed real.

Question 14

Tags: Transformation of Graphs · Difficulty: 7.5

The following sequence of transformations is applied, in the order listed, to the curve $y = 2x^2 - 8x + 11$: (1) translation by $\binom{-1}{4}$; (2) reflection in the line $y = 2$; (3) stretch parallel to the y-axis with scale factor 3; (4) stretch parallel to the x-axis with scale factor $\frac{1}{2}$. What is the equation of the resulting curve?

A $y = -24x^2 + 24x - 15$

B $y = -24x^2 + 24x - 27$

C $y = 24x^2 - 24x + 27$

D $y = -6x^2 + 12x - 15$

E $y = -8x^2 + 8x - 5$

F $y = -24x^2 + 72x - 63$

G $y = -8x^2 + 8x - 15$

H $y = -8x^2 + 6x - 5$

Solution 14

Answer: A

Start by writing the curve in vertex form:

$$y = 2x^2 - 8x + 11 = 2(x - 2)^2 + 3,$$

so the initial vertex is $(2, 3)$. Track the vertex through each transformation.

Step 1. Translation by $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ shifts the vertex by $(-1, +4)$:

$$(2, 3) \longrightarrow (1, 7).$$

Step 2. Reflection in the line $y = 2$ sends y to $4 - y$:

$$(1, 7) \longrightarrow (1, 4 - 7) = (1, -3).$$

Step 3. Stretch parallel to the y-axis with scale factor 3 multiplies the y-coordinate by 3:

$$(1, -3) \longrightarrow (1, -9).$$

Step 4. Stretch parallel to the x-axis with scale factor $\frac{1}{2}$ multiplies the x-coordinate by $\frac{1}{2}$:

$$(1, -9) \longrightarrow \left(\frac{1}{2},\ -9\right).$$

Step 5. Track the leading coefficient a in $y = a(x - h)^2 + k$. Translation preserves a. Reflection in $y = 2$ flips the sign: $a = 2 \to -2$. The y-stretch by 3 multiplies a by 3: $-2 \to -6$. The x-stretch by $\frac{1}{2}$ replaces $(x - h)$ with $(2x - h) = 2\left(x - \frac{h}{2}\right)$ inside the square, multiplying a by 4: $-6 \to -24$. The resulting curve is

$$y = -24\left(x - \frac{1}{2}\right)^2 - 9 = -24x^2 + 24x - 15,$$

confirming the vertex $\left(\frac{1}{2},\ -9\right)$. The answer is A.

Question 15

Tags: General Trigonometry · Difficulty: 7.5

Find the sum of the solutions of the equation

$$\sqrt{1-\sin^2 x} = 2\sin^2 x - \cos x$$

where $0 \le x \le 360°$.

A 180°

B 360°

C 540°

D 720°

E 900°

Solution 15

Answer: C

The left-hand side is $\sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. Using $\sin^2 x = 1 - \cos^2 x$, the right-hand side becomes $2(1-\cos^2 x) - \cos x = 2 - 2\cos^2 x - \cos x$. Let $c = \cos x$, so the equation is $|c| = 2 - 2c^2 - c$.

Case 1: $c \ge 0$ (so $x \in [0°, 90°] \cup [270°, 360°]$). Then $c = 2 - 2c^2 - c$, i.e. $c^2 + c - 1 = 0$, giving $c = \frac{-1\pm\sqrt{5}}{2}$. The minus root is ≈ -1.618, outside $[-1, 1]$ and violating $c \ge 0$, so reject. The plus root $c = \frac{\sqrt{5}-1}{2} \approx 0.618$ is admissible. On $[0°, 360°]$ the equation $\cos x = \frac{\sqrt{5}-1}{2}$ has two solutions, $x = \alpha$ and $x = 360° - \alpha$ where $\alpha = \arccos\left(\frac{\sqrt{5}-1}{2}\right)$, summing to 360°.

Case 2: $c < 0$ (so $x \in (90°, 270°)$). Then $-c = 2 - 2c^2 - c$, i.e. $2c^2 = 2$, so $c = \pm 1$. Only $c = -1$ satisfies $c < 0$, giving $x = 180°$. Check: LHS $= 1$, RHS $= 2(0) - (-1) = 1$. ✓

Total: 3 solutions. Sum $= 360° + 180° = 540°$. The answer is C.

Question 16

Tags: Exponentials and Logarithms · Difficulty: 7.5

Find the maximum value of the function

$$\mathrm{f}(x) = \frac{1}{4^x + 4^{-x} - 2\left(2^x + 2^{-x}\right) + 8}.$$

A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{4}$

E 5

F 6

Solution 16

Answer: B

Let $u = 2^x + 2^{-x}$. Writing $t = 2^x > 0$, we have $u = t + \frac{1}{t}$, so by AM-GM $u \geq 2$ with equality at $t = 1$ (i.e. $x = 0$); as $t \to 0^+$ or $t \to \infty$, $u \to \infty$, so u ranges over $[2, \infty)$.

Note that $u^2 = 4^x + 2 + 4^{-x}$, hence $4^x + 4^{-x} = u^2 - 2$. The denominator becomes

$$\left(u^2 - 2\right) - 2u + 8 = u^2 - 2u + 6 = (u-1)^2 + 5.$$

The **unconstrained** minimum of $(u-1)^2 + 5$ is 5 at $u = 1$, but $u = 1$ is **not achievable** since $u \geq 2$. On $[2, \infty)$ the function $u^2 - 2u + 6$ is increasing (its vertex $u = 1$ lies to the left of the interval), so its minimum on the achievable range is at $u = 2$, giving $(2-1)^2 + 5 = 6$. This is attained at $x = 0$.

Hence the denominator's minimum value is 6, and the maximum of f is $\frac{1}{6}$.

Question 17

Tags: Circle Geometry, Ratio and Proportion · Difficulty: 7.5

The circle C_1 has equation $(x-1)^2 + y^2 = 4$. The circle C_2 has radius 5 and centre (a, b), where (a, b) is chosen uniformly at random from the rectangle $-5 \leq a \leq 5$, $-3 \leq b \leq 3$.

What is the probability that the **circumferences** of C_1 and C_2 meet in at least one point?

A $\frac{3\pi}{20}$

B $\frac{20-3\pi}{20}$

C $\frac{60-49\pi}{60}$

D $\frac{49\pi-9\pi}{60}$

E 1

F $\frac{20-9\pi}{20}$

Solution 17

Answer: B

Let d be the distance from (a, b) to the centre of C_1, namely $(1, 0)$, so $d = \sqrt{(a-1)^2 + b^2}$. The radii are $r_1 = 2$ and $r_2 = 5$, so C_1 and C_2 intersect iff $|r_1 - r_2| \leq d \leq r_1 + r_2$, i.e. $3 \leq d \leq 7$.

Outer bound is automatic. The point in the rectangle farthest from $(1, 0)$ is at a corner $(-5, \pm 3)$, giving $d = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5} \approx 6.71 < 7$. So $d \leq 7$ holds throughout the rectangle.

Inner bound is active. The set $d < 3$ is the open disk of radius 3 centred at $(1, 0)$. This disk has $x \in [-2, 4] \subset [-5, 5]$ and $y \in [-3, 3]$, so it lies inside the rectangle (touching the edges $y = \pm 3$ only at the single points $(1, \pm 3)$, which have measure zero). Its area is 9π.

Probability. The rectangle has area $10 \times 6 = 60$. The favourable region (where C_1 and C_2 intersect) has area $60 - 9\pi$. Hence

$$P = \frac{60 - 9\pi}{60} = \frac{20 - 3\pi}{20}.$$

Question 18

Tags: General Trigonometry, Inequalities · Difficulty: 7.5

Let $0 \le x \le 2\pi$. Find the total length of the intervals on which all three of the following inequalities hold simultaneously

$$\sin x \ge \frac{1}{2}, \qquad \cos 2x \le 0, \qquad \tan x \le \sqrt{3}.$$

A $\frac{\pi}{12}$

B $\frac{\pi}{6}$

C $\frac{\pi}{4}$

D $\frac{\pi}{3}$

E $\frac{5\pi}{12}$

F $\frac{\pi}{2}$

G $\frac{7\pi}{12}$

H $\frac{2\pi}{3}$

Solution 18

Answer: D

Solve each inequality on $[0, 2\pi]$.

(1) $\sin x \ge \frac{1}{2}$ holds on $[\frac{\pi}{6}, \frac{5\pi}{6}]$.

(2) $\cos 2x \le 0$. With $\theta = 2x \in [0, 4\pi]$, $\cos\theta \le 0$ on $[\frac{\pi}{2}, \frac{3\pi}{2}] \cup [\frac{5\pi}{2}, \frac{7\pi}{2}]$, so $x \in [\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}]$.

(3) $\tan x \le \sqrt{3}$. Branch by branch (excluding $\frac{\pi}{2}, \frac{3\pi}{2}$ where tan is undefined):

$$\begin{aligned}
[0, \frac{\pi}{2}) &: \quad \tan x \le \sqrt{3} \text{ on } [0, \frac{\pi}{3}], \\
(\frac{\pi}{2}, \pi] &: \quad \tan x \le 0, \text{ always}, \\
[\pi, \frac{3\pi}{2}) &: \quad \tan x \le \sqrt{3} \text{ on } [\pi, \frac{4\pi}{3}], \\
(\frac{3\pi}{2}, 2\pi] &: \quad \tan x \le 0, \text{ always}.
\end{aligned}$$

So $\tan x \le \sqrt{3}$ on $[0, \frac{\pi}{3}] \cup (\frac{\pi}{2}, \frac{4\pi}{3}] \cup (\frac{3\pi}{2}, 2\pi]$.

Intersect (1) and (2): $[\frac{\pi}{6}, \frac{5\pi}{6}] \cap ([\frac{\pi}{4}, \frac{3\pi}{4}] \cup [\frac{5\pi}{4}, \frac{7\pi}{4}]) = [\frac{\pi}{4}, \frac{3\pi}{4}]$ (the second piece is disjoint from $[\frac{\pi}{6}, \frac{5\pi}{6}]$).

Intersect with (3): $[\frac{\pi}{4}, \frac{3\pi}{4}]$ meets the tangent set in two pieces:

$$[\frac{\pi}{4}, \frac{3\pi}{4}] \cap [0, \frac{\pi}{3}] = [\frac{\pi}{4}, \frac{\pi}{3}], \quad \text{length } \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12};$$

$$[\frac{\pi}{4}, \frac{3\pi}{4}] \cap (\frac{\pi}{2}, \frac{4\pi}{3}] = (\frac{\pi}{2}, \frac{3\pi}{4}], \quad \text{length } \frac{3\pi}{4} - \frac{\pi}{2} = \frac{\pi}{4}.$$

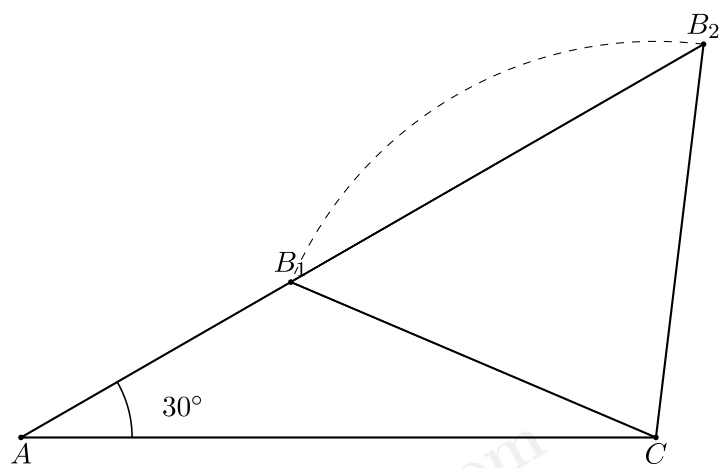
Total length $= \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{12} + \frac{3\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3}$.

Answer: **D**.

Question 19

Tags: General Trigonometry, Geometry · Difficulty: 8

In a triangle ABC the angle at vertex A is $30°$. The side BC (opposite to A) has length $(x-2)(x-3)$ and the side AC (one of the sides containing A) has length $(3-x)(x-8)$. Find the complete set of values of x for which there are two non-congruent triangles with these data.



A $3 < x < 4$

B $3 < x < 5$

C $3 < x < 8$

D $4 < x < 5$

E $4 < x < 8$

F $5 < x < 8$

Solution 19

Answer: D

Let $a = BC$ be the side opposite the $30°$ angle, and let $b = AC$ be the adjacent side. Then

$$a = (x-2)(x-3), \qquad b = (3-x)(x-8) = (x-3)(8-x).$$

Both side lengths must be positive. The condition $a > 0$ gives $x < 2$ or $x > 3$, while $b > 0$ gives $3 < x < 8$. Hence the possible values must first satisfy

$$3 < x < 8.$$

This is the ambiguous SSA case: one angle, the opposite side, and one adjacent side are given. For there to be two non-congruent triangles, the opposite side must be longer than the height but shorter than the adjacent side:

$$b \sin 30^\circ < a < b.$$

Since $\sin 30^\circ = \frac{1}{2}$, this becomes

$$\frac{b}{2} < a < b.$$

Now substitute $a = (x-3)(x-2)$ and $b = (x-3)(8-x)$. Since $3 < x < 8$, we have $x - 3 > 0$.

First,

$$a < b$$

gives

$$(x-3)(x-2) < (x-3)(8-x).$$

Dividing by $x - 3 > 0$ gives

$$x - 2 < 8 - x,$$

so

$$x < 5.$$

Second,

$$\frac{b}{2} < a$$

gives

$$(x-3)(8-x) < 2(x-3)(x-2).$$

Dividing by $x - 3 > 0$ gives

$$8 - x < 2x - 4,$$

so

$$x > 4.$$

Combining these conditions with $3 < x < 8$ gives

$$4 < x < 5.$$

Therefore the correct answer is D.

Question 20

Tags: Sequences and Series, General Algebra · Difficulty: 8.5

A sequence of real numbers (a_n) is defined by $a_1 = 2$ and the recurrence

$$a_{n+1}(a_n - 1) = \frac{1}{2}a_n^2 - a_n + \frac{1}{2} \qquad (n \geq 1).$$

Find the value of

$$\sum_{n=1}^{\infty}(a_n + 1).$$

A $\frac{3}{2}$

B 4

C $\frac{5}{2}$

D 6

E 8

F $\frac{9}{4}$

G $\frac{5}{4}$

H ∞

Solution 20

Answer: D

Step 1: simplify the recurrence. The right-hand side factorises as

$$\frac{1}{2}a_n^2 - a_n + \frac{1}{2} = \frac{1}{2}(a_n - 1)^2.$$

Provisionally dividing both sides by $a_n - 1$ (we verify validity at the end):

$$a_{n+1} = \frac{1}{2}(a_n - 1).$$

Step 2: convert to a geometric progression. Add 1 to both sides:

$$a_{n+1} + 1 = \frac{1}{2}(a_n - 1) + 1 = \frac{1}{2}(a_n + 1).$$

Let $b_n = a_n + 1$. Then $b_{n+1} = \frac{1}{2}b_n$ with $b_1 = a_1 + 1 = 3$, so (b_n) is geometric with first term 3 and common ratio $\frac{1}{2}$. Hence $b_n = 3 \cdot \left(\frac{1}{2}\right)^{n-1} = \frac{3}{2^{n-1}}$, and $a_n = \frac{3}{2^{n-1}} - 1$. The equation $a_n = 1$ would require $2^{n-1} = \frac{3}{2}$, which has no integer solution, so $a_n \neq 1$ for all n and the division in Step 1 is valid throughout.

Step 3: sum the geometric series. Since $|r| = \frac{1}{2} < 1$, the series converges:

$$\sum_{n=1}^{\infty}(a_n + 1) \;=\; \sum_{n=1}^{\infty} b_n \;=\; \frac{3}{1 - \frac{1}{2}} \;=\; 6.$$

The answer is D.