# TMUA Worked Examples Worksheet 1 Solutions

**Question 1**

The first attempt shows that the password has none of the letters in the positions used in *abcd*. The second attempt rules out the positions used in *bcda*.
For the first position, the letter must therefore be either $c$ or $d$.
If the first letter is $c$, the remaining restrictions force the password to be *cdab*. If the first letter is $d$, they force the password to be *dabc*. Therefore there are exactly two possible passwords:

$$cdab \quad \text{and} \quad dabc.$$

The password cannot yet be identified, so at least one further attempt is required. The player can now attempt *cdab*. A response of 4 means that the password is *cdab*, while a response of 0 means that it is *dabc*.
Hence one further attempt is both necessary and sufficient.
The answer is **B**.

**Question 2**

The four conditions are nested as follows:

$$0 < x < 1 \;\Rightarrow\; 0 < x < 2 \;\Rightarrow\; -1 < x < 2 \;\Rightarrow\; -2 < x < 3.$$

Suppose any condition other than $-2 < x < 3$ were necessary for $P$. Since that condition implies $-2 < x < 3$, the condition $-2 < x < 3$ would also be necessary. There would then be at least two necessary conditions among the options, contradicting the information given.
Therefore $-2 < x < 3$ must be the unique necessary condition. It is not sufficient because none of the listed conditions is equivalent to $P$.
The answer is **C**.

**Question 3**

The four conditions are nested as follows:

$$0 < x < 1 \;\Rightarrow\; 0 < x < 2 \;\Rightarrow\; -1 < x < 3 \;\Rightarrow\; -2 < x < 4.$$

Suppose any condition other than $0 < x < 1$ were sufficient for $Q$. Every narrower condition would then also be sufficient for $Q$, giving at least two sufficient conditions among the options. This contradicts the information given.
Therefore $0 < x < 1$ must be the unique sufficient condition. It is not necessary because none of the listed conditions is equivalent to $Q$.
The answer is **D**.

**Question 4**

Statements C and D are opposites, so exactly one of them must be true.
Since exactly one of all four statements is true, statements A and B must both be false. The negation of A gives

$$x \leq 0,$$

while the negation of B gives

$$x + y \geq 0.$$

Therefore

$$y \geq -x.$$

Since $x \leq 0$, we have $-x \geq 0$, and hence

$$y \geq 0.$$

It follows that

$$x \leq 0 \leq y,$$

so $x \leq y$. Thus statement D is true.
The answer is **D**.

**Question 5**

Rewrite each statement in the form: condition on $x$ and $b$ implies condition on $g(x)$.

$$\begin{aligned}
\textbf{A:}\quad & x < b \implies g(x) \geq 0.\\
\textbf{B:}\quad & x < b \iff g(x) \geq 0.\\
\textbf{C:}\quad & x < b \implies g(x) \geq 0.\\
\textbf{D:}\quad & x < b \implies g(x) \geq 0.\\
\textbf{E:}\quad & x < b \implies g(x) \geq 0.\\
\textbf{F:}\quad & x \geq b \implies g(x) \leq 0.
\end{aligned}$$

Therefore A, C, D and E are equivalent. Since exactly one statement is true, this group of equivalent statements must be false.
Also, B implies A, so B cannot be the unique true statement.
Thus, if exactly one statement is true, it must be F.
The answer is **F**.

### Question 6

Rewrite each statement with the condition on $x$ first, using the contrapositive where needed.

$$\begin{aligned}
&\textbf{A:} && |x| < 2 \implies Q.\\
&\textbf{B:} && |x| \leq 2 \implies Q.\\
&\textbf{C:} && x \leq 1 \text{ or } x \geq 4 \implies \neg Q.\\
&\textbf{D:} && |x| < 3 \implies Q.\\
&\textbf{E:} && x \leq 1 \text{ or } x \geq 4 \implies \neg Q.\\
&\textbf{F:} && |x| \geq 2 \iff \neg Q.
\end{aligned}$$

Statements C and E are equivalent, so neither can be the only true statement.
Statement D implies B, since $|x| \leq 2$ is contained inside $|x| < 3$. Another way to understand this is that D says whenever $x$ is a number between $-3$ and 3, $Q$ is true. Given this is the case, B is also true, and hence D implies B.
Statement B implies A, since $|x| < 2$ is contained inside $|x| \leq 2$. Therefore B and D cannot be the only true statement.
Statement F implies

$$|x| < 2 \implies Q,$$

so F implies A. Therefore F cannot be the only true statement.
Thus the only statement which can be true by itself is A.
The answer is **A**.

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**Question 7**

For two non-congruent triangles to be possible, the side $BC$ must lie strictly between the perpendicular height and the side $AB$.
The perpendicular height is

$$AB\sin 30^\circ = 2x^2 \cdot \frac{1}{2} = x^2.$$

Therefore, two non-congruent triangles are possible if and only if

$$x^2 < 3x < 2x^2.$$

Since $x > 0$, the first inequality gives

$$x < 3,$$

and the second inequality gives

$$x > \frac{3}{2}.$$

Hence

$$\frac{3}{2} < x < 3.$$

It follows that

$$1 < x < 4,$$

so statement A can be deduced.
The other statements cannot be deduced. For example, $x = \frac{5}{2}$ shows that B need not be true, $x = \frac{7}{4}$ shows that C need not be true, and $x = \frac{11}{4}$ shows that D need not be true.
Notice that $1 < x < 4$ is necessary but not sufficient. For example, $x = \frac{5}{4}$ satisfies $1 < x < 4$, but does not give two non-congruent triangles.
The answer is **A**.

**Question 8**

By the fewest statement checks available to a lucky student, we mean the smallest number of statements that could be checked before the results uniquely determine one of the four answer choices. We may therefore assume that the actual combination of true and false statements is whichever is most favourable to the student.

In A, suppose that all three statements are true. If the student checks Statement 3 and finds that it is true, only one answer choice remains possible. Thus A can be solved with just one check.

In B, C and D, each individual statement is true in two answer choices and false in two answer choices. Therefore one check always leaves two possibilities. However, in each case, a suitable pair of statements gives the four different outcomes

$$FF, \quad FT, \quad TF, \quad TT,$$

so two checks are sufficient.

Therefore the minimum numbers of checks for A, B, C and D are

$$1, \quad 2, \quad 2, \quad 2.$$

The student should therefore choose A.

The answer is **A**.

*Unfortunately, designing a TMUA question to minimise the amount of TMUA required is logical enough that it may itself be assessed by TMUA. :)*

# TMUA Worked Examples Worksheet 1 Solutions

**Question 9**

We may assume that Statements 1 and 2 are true, while Statements 3 and 4 are false. The correct answer option would therefore be

Statements 1 and 2 only are true.

To force the student to check Statement 1, there must be an incorrect option that differs from the correct option only in the truth value of Statement 1. Similarly, there must be such an option for each of Statements 2, 3 and 4.

For example, the setter could use the following five options:

A Statements 1 and 2 only are true.
B Statement 2 only is true.
C Statement 1 only is true.
D Statements 1, 2 and 3 only are true.
E Statements 1, 2 and 4 only are true.

Option B forces the student to check Statement 1, option C forces the student to check Statement 2, option D forces the student to check Statement 3, and option E forces the student to check Statement 4.

Thus, five options are sufficient.

They are also necessary. Each of the four statements requires a different incorrect option that agrees with the correct option on the other three statements but differs on that particular statement. Therefore, four incorrect options and one correct option are required.

The answer is **A**.

*Now you know what I think about all day—how best to trip up students! :)*

**Question 10**

Suppose, for a contradiction, that $A$ is a liar.
Then $D$'s statement that $A$ is a liar is true, so $D$ must be a saint.
Since $D$ is a saint, $D$'s statement that $C$ is a saint must be true. Therefore, $C$ is a saint.
Since $C$ is a saint, $C$'s statement that $B$ is a saint must be true. Therefore, $B$ is also a saint.
However, $B$ states that $A$ and $C$ are either both saints or both liars. This is false, since $A$ is a liar and $C$ is a saint. This contradicts the conclusion that $B$ is a saint.
Therefore, $A$ cannot be a liar, so $A$ is a saint.
It follows that all of $A$'s statements are true. In particular, exactly one of the four students is a saint. Hence $B$, $C$ and $D$ are all liars.
This assignment is consistent with every statement made:

$B$'s claim that $A$ and $C$ have the same status is false,
$B$'s claim that $D$ is a saint is false,
$C$'s claim that $B$ is a saint is false,
$C$'s claim that $A$ and $D$ have the same status is false,
$D$'s claim that $A$ is a liar is false,
$D$'s claim that $C$ is a saint is false.

Therefore, $A$ is the only saint. It can also be shown by contradiction that none of $B$, $C$ and $D$ can be the saint.
The answer is **A**.

*Well done! You have completed my logic course, survived my torturous questions, and been forcibly amused by my Easter egg questions. You have finally truly earned yourself one very delicious imaginary biscuit. – Joe :)*