Proof by Contradiction Worksheet 1 Solutions

Question 1

Proof: Suppose there exist positive integers x and y satisfying

$$4x^2 - y^2 = 9.$$

Now let's see what can be deduced.
$\Rightarrow (2x - y)(2x + y) = 9$.
$\Rightarrow 2x - y > 0$, since $2x + y$ is positive.
$\Rightarrow 2x + y = 9$ and $2x - y = 1$, since $2x + y > 2x - y > 0$, and the only possible factorisation of 9 with two different positive factors is 9 and 1.
$\Rightarrow$ Adding the equations gives $4x = 10$, or $x = 2.5$.
However, this cannot be right, since x is supposed to be a positive integer. Every time we assume that such positive integers exist, we reach this contradiction. Therefore, no positive integers x and y satisfy $4x^2 - y^2 = 9$, and this concludes the proof.

Question 2

Proof: Suppose Hugo lives on floor 2. Let the number of residents on each floor be f_1, f_2, f_3 and f_4.
$\Rightarrow f_1 = 9$, since 9 people live below Hugo.
Zane has fewer people living above him than Eva has above her.
$\Rightarrow$ Zane must live above Eva, and neither can live on floor 4.
$\Rightarrow$ Zane and Eva must live on floors 3 and 1 respectively.
$\Rightarrow f_4 = 17$, since there are 17 people above Zane, and $f_2 + f_3 + f_4 = 22$, since there are 22 people above Eva.
$\Rightarrow f_2 + f_3 = 5$, obtained by substituting $f_4 = 17$.
$\Rightarrow$ Maya must live on floor 4.
$\Rightarrow 9 + f_2 + f_3 = 18$, since there are 18 people below Maya.
$\Rightarrow f_2 + f_3 = 9$.
However, this cannot be right, since we already found that $f_2 + f_3 = 5$.
Therefore, the assumption that Hugo lives on floor 2 is wrong. Hence Hugo cannot live on floor 2, and this concludes the proof.
Remark: It is possible to continue and deduce precisely who lives on each floor, as well as the number of residents on each floor.

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Question 3

Proof: Suppose there is a smallest positive rational number, and call it r. Now let's see what can be deduced.
$\Rightarrow r/2$ is rational, since a rational number divided by the non-zero integer 2 is rational.
$\Rightarrow r/2 > 0$.
$\Rightarrow r/2 < r$.
Therefore, $r/2$ is a positive rational number smaller than r.
However, this cannot be right, because r was assumed to be the smallest positive rational number. Every time we assume that a smallest positive rational number exists, we reach this contradiction. Therefore, there is no smallest positive rational number, and this concludes the proof.

Question 4

Proof: Suppose that, among a group of 13 people, no two were born in the same month. Now let's see what can be deduced.
$\Rightarrow$ Each of the 13 people must have a different birth month.
$\Rightarrow$ There must be at least 13 different months in the year.
However, this cannot be right, because a year has only 12 months. Every time we assume that no two of the 13 people share a birth month, we reach this contradiction. Therefore, at least two of them were born in the same month, and this concludes the proof.

Question 5

Proof: Suppose there exist integers a, b and k satisfying

$$a^2 + b^2 = 4k + 3.$$

For any integer t, either t is even or t is odd.
If $t = 2m$, then

$$t^2 = 4m^2,$$

so t^2 is a multiple of 4.
If $t = 2m + 1$, then

$$t^2 = 4m(m + 1) + 1,$$

so t^2 is one more than a multiple of 4.
$\Rightarrow$ Each of a^2 and b^2 is either a multiple of 4 or one more than a multiple of 4.
$\Rightarrow a^2 + b^2$ is either a multiple of 4, one more than a multiple of 4, or two more than a multiple of 4.
However, this cannot be right, because $4k + 3$ is three more than a multiple of 4. Every time we assume that such integers exist, we reach this contradiction. Therefore, no integers a, b and k satisfy $a^2 + b^2 = 4k + 3$, and this concludes the proof.

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Question 6

We rephrase the statement as

$$\text{given that } n^2 \text{ is divisible by 3, prove that } n \text{ is divisible by 3.}$$

Proof: Suppose we are given that n^2 is divisible by 3, and assume for contradiction that n is not divisible by 3.
$\Rightarrow n = 3k+1$ or $n = 3k+2$ for some integer k.
If $n = 3k+1$, then

$$n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

If $n = 3k+2$, then

$$n^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1.$$

$\Rightarrow$ In either case, n^2 is not divisible by 3.
However, this cannot be right, since we were given that n^2 is divisible by 3. Every time we assume that n is not divisible by 3, we reach this contradiction. Therefore, n is divisible by 3, and this concludes the proof.

Question 7

We rephrase the statement as

$$\text{given that } a+b \text{ is odd, prove that exactly one of } a \text{ and } b \text{ is odd.}$$

Proof: Suppose we are given that $a+b$ is odd, and assume for contradiction that it is not true that exactly one of a and b is odd.
$\Rightarrow$ Either a and b are both even, or a and b are both odd.
If both are even, then $a+b$ is even.
If both are odd, then $a+b$ is also even.
$\Rightarrow$ In either case, $a+b$ is even.
However, this cannot be right, since we were given that $a+b$ is odd. Therefore, exactly one of a and b is odd, and this concludes the proof.

Question 8

We rephrase the statement as

$$\text{given that } x+y > 12, \text{ prove that } x > 6 \text{ or } y > 6.$$

Proof: Suppose we are given that $x+y > 12$, and assume for contradiction that $x > 6$ or $y > 6$ is false.
$\Rightarrow x \leq 6$ and $y \leq 6$.
$\Rightarrow x + y \leq 12$.
However, this cannot be right, since we were given that $x+y > 12$. Therefore, $x > 6$ or $y > 6$, and this concludes the proof.

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Question 9

We rephrase the statement as

given that r is rational and s is irrational, prove that $r + s$ is irrational.

Proof: Suppose we are given that r is rational and s is irrational, and assume for contradiction that $r + s$ is rational.
Since r and $r + s$ are supposedly rational, there exist integers a, b, c, d, where $b \neq 0$ and $d \neq 0$, such that

$$r = \frac{a}{b} \qquad \text{and} \qquad r + s = \frac{c}{d}.$$

Therefore,

$$s = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}.$$

Since $bc - ad$ and bd are integers, and $bd \neq 0$, this shows that s is rational.
However, this cannot be right, since we were given that s is irrational. Therefore, our assumption that $r + s$ is rational must be false. Hence $r + s$ is irrational, and this concludes the proof.

Question 10

We rephrase the statement as

given that $c^2 > a^2 + b^2$, prove that C is obtuse.

Proof: Suppose we are given that $c^2 > a^2 + b^2$, and assume for contradiction that C is not obtuse.
$\Rightarrow C \leq 90°$.
$\Rightarrow \cos C \geq 0$.
By the cosine rule,

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

$\Rightarrow c^2 \leq a^2 + b^2$.
However, this cannot be right, since we were given that $c^2 > a^2 + b^2$. Therefore, C is obtuse, and this concludes the proof.

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Question 11

Proof: Suppose, for contradiction, that every student in the class has mutually complained about the Easter egg question with a different number of classmates. Let there be n students in the class. Each student can have mutually complained with between 0 and $n-1$ classmates.
$\Rightarrow$ There are n students whose numbers of complaint partners are n different integers chosen from $0, 1, \ldots, n-1$.
$\Rightarrow$ These numbers must be exactly $0, 1, \ldots, n-1$.
$\Rightarrow$ One student has mutually complained with no classmates, while another student has mutually complained with all $n-1$ classmates.
However, this cannot be right. The student who has mutually complained with all $n-1$ classmates must have mutually complained with the student who has mutually complained with no classmates.
$\Rightarrow$ It is impossible for every student in the class to have mutually complained about the Easter egg question with a different number of classmates, and this concludes the proof.
It is perhaps best, then, for every student to complain about the Easter egg question exactly zero times! :)