Converse and Equivalence Worksheet 1 Solutions

Question 1

The phrase P if and only if Q means that both directions hold:

$$P \Rightarrow Q \qquad \text{and} \qquad Q \Rightarrow P.$$

Option A states exactly these two implications.
The answer is **A**.

Question 2

The phrase "exactly when" indicates an equivalence. Therefore the door opens only when the silver key is inserted, and inserting the silver key also guarantees that the door opens. Option A gives both directions.
The answer is **A**.

Question 3

The phrase "a square only if it is a rectangle" means that being a rectangle is necessary for being a square. This is exactly the implication

$$\text{square} \Rightarrow \text{rectangle}.$$

The other options either reverse the implication or make a different claim.
The answer is **C**.

Question 4

An integer is a multiple of both 3 and 4 exactly when it is divisible by their lowest common multiple. Since

$$\text{lcm}(3, 4) = 12,$$

this is equivalent to the integer being a multiple of 12. The other conditions describe different sets of integers.
The answer is **A**.

Question 5

If n is even, then $n = 2k$ for some integer k, so $n^2 = 4k^2$ is a multiple of 4. Conversely, if n were odd, then n^2 would also be odd and therefore could not be a multiple of 4. Hence the two conditions are equivalent.
The answer is **B**.

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Question 6

The product xy is positive exactly when x and y are both positive or both negative. Requiring $x \neq 0$ and $y \neq 0$ excludes zero, while

$$x > 0 \quad \text{if and only if} \quad y > 0$$

ensures that x and y have the same sign.
The answer is **D**.

Question 7

The inequality $|x - 2| \leq 3$ means that the distance between x and 2 is at most 3. Hence

$$-3 \leq x - 2 \leq 3 \Leftrightarrow -1 \leq x \leq 5.$$

The answer is **A**.

Question 8

Completing the square gives

$$x^2 + 2x + 9 = (x + 1)^2 + 8.$$

Since $(x + 1)^2 \geq 0$, the expression is at least 8 and is therefore positive for every real x.
The answer is **A**.

Question 9

The fraction changes sign at $x = -1$ and $x = 4$. It is positive when the numerator and denominator have the same sign, which occurs when

$$x < -1 \qquad \text{or} \qquad x > 4.$$

Neither endpoint is included: the expression is undefined at $x = -1$ and equals zero at $x = 4$.
The answer is **A**.

Question 10

Rearranging and factoring gives

$$x^2 = 5x \Leftrightarrow x^2 - 5x = 0 \Leftrightarrow x(x - 5) = 0.$$

By the zero-product property, this is equivalent to $x = 0$ or $x = 5$.
The answer is **A**.

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Question 11

A fraction equals zero exactly when its numerator is zero and its denominator is non-zero. Here $x-3=0$ gives $x=3$, and this does not make $x+2$ zero. The value $x=-2$ is excluded because the fraction is undefined there.
The answer is **A**.

Question 12

The numerator factors as

$$x^2-1=(x-1)(x+1).$$

The fraction is defined only when $x\neq 1$. Its numerator is zero at $x=1$ or $x=-1$, but $x=1$ is excluded. Therefore the original equation is equivalent to $x=-1$.
The answer is **C**.

Question 13

Let $u=|x|$, so $u\geq 0$. Since $x^2=|x|^2=u^2$, the equation becomes

$$u^2=u \Leftrightarrow u(u-1)=0.$$

Thus $|x|=0$ or $|x|=1$, giving $x=-1$, $x=0$, or $x=1$.
The answer is **A**.

Question 14

The left-hand side is non-negative, so any solution must satisfy $x\geq 0$. Squaring gives

$$\sqrt{x+6}=x \Rightarrow x+6=x^2 \Leftrightarrow (x-3)(x+2)=0.$$

The candidates are $x=3$ and $x=-2$, but $x=-2$ does not satisfy the original equation. Hence $x=3$ is the only solution.
The answer is **A**.

Question 15

For $x\neq 0$,

$$\frac{2}{x}<x+1 \Leftrightarrow \frac{x^2+x-2}{x}>0 \Leftrightarrow \frac{(x-1)(x+2)}{x}>0.$$

Multiplying this expression by the positive quantity x^2 does not change its sign, so it has the same solution set as

$$x(x-1)(x+2)>0.$$

Both conditions exclude $x=0$.
The answer is **A**.

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Question 16

By De Morgan's law,

$$(A^c \cup B)^c = (A^c)^c \cap B^c = A \cap B^c.$$

This is the set of elements that are in A and not in B.
The answer is **A**.

Question 17

A rhombus is a quadrilateral with four equal sides. Therefore the given condition is exactly the defining condition for the quadrilateral to be a rhombus. A square is a special type of rhombus, while a general rhombus need not be a square or a rectangle.
The answer is **C**.

Question 18

Since c is the longest side, the converse of Pythagoras' theorem gives

$$a^2 + b^2 < c^2$$

exactly when the angle opposite c is obtuse. Equality gives a right angle, while $a^2 + b^2 > c^2$ gives an acute triangle.
The answer is **C**.

Question 19

Congruent parallelograms have equal corresponding side lengths and equal corresponding angles, so congruence implies option A. Conversely, $AB = EF$, $BC = FG$ and $\angle ABC = \angle EFG$ give two corresponding adjacent sides and their included angle. These determine the whole parallelogram, so the two parallelograms are congruent. Thus option A is equivalent to congruence.
The answer is **A**.

Question 20

The original statement says that it is not true that every student both understands Question 20 and takes it seriously.
Therefore, at least one student must fail to satisfy one of these two conditions. That student either does not understand Question 20 or does not take it seriously.
The answer is **A**.
Understanding Question 20 and taking it seriously are entirely separate matters, but both are equally important. :)